A FUNCTORIAL CONSTRUCTION OF MODULI OF SHEAVES

LUIS ÁLVAREZ-CÓNSUL AND ALASTAIR KING

Dedicated to the memory of Joseph Le Potier.

ABSTRACT. We show how natural functors from the category of coherent sheaves on a projective scheme to categories of Kronecker modules can be used to construct moduli spaces of semistable sheaves. This construction simplifies or clarifies technical aspects of existing constructions and yields new simpler definitions of theta functions, about which more complete results can be proved.

1. INTRODUCTION

Let X be a projective scheme over an algebraically closed field of arbitrary characteristic. An important set of invariants of X are the projective schemes $\mathcal{M}_X^{ss}(P)$, which are the moduli spaces for semistable coherent sheaves of $\mathcal{O}_X$-modules with fixed Hilbert polynomial P, with respect to a very ample invertible sheaf $\mathcal{O}(1)$.

Indeed, it has been a fundamental problem to define and construct these moduli spaces in this generality, ever since Mumford [24] and Seshadri [31] did so for smooth projective curves, introducing the notions of stability, semistability and S-equivalence of vector bundles. Gieseker [8] and Maruyama [21] extended the definitions and constructions to torsion-free sheaves on higher dimensional smooth projective varieties. Simpson [34] completed the programme by extending to 'pure' sheaves on arbitrary projective schemes. Langer [16] showed that the constructions can be carried out in arbitrary characteristic.

Since the beginning, the method of construction has been to identify isomorphism classes of sheaves with orbits of a reductive group acting on a certain Quot-scheme and then apply Geometric Invariant Theory (GIT), as developed by Mumford [25] for precisely such applications. Thus one is required to find a projective embedding of this Quot-scheme, with a natural linearisation of the group action, so that

the semistable sheaves correspond to GIT-semistable orbits. One of the most natural projective embeddings to use, as Simpson [34] does, is into the Grassmannian originally used by Grothendieck [10] to construct the Quot-scheme. Thus, at least in characteristic zero, the moduli space $\mathcal{M}_X^{ss}(P)$ may be realised as a closed subscheme of a GIT quotient of this Grassmannian. (In characteristic p, the embedding is set-theoretic, but not necessarily scheme-theoretic.)

The observation from which this paper grew is that the GIT quotient of this Grassmannian has a natural moduli interpretation (see Remark 4.9) as a moduli space of (generalised) Kronecker modules or, equivalently, modules for a certain finite dimensional associative algebra A. Furthermore, the construction itself can be described in terms of a natural functor from $\mathcal{O}_X$-modules to A-modules.

Taking this functorial point of view, many of the more technical aspects become much clearer and lead to a 'one-step' construction of the moduli spaces which is conceptually simpler than (although structurally parallel to) the 'two-step' process through the Quot-scheme and the Grothendieck-Simpson embedding.

Let us begin by reviewing this two-step process. For the first step, one chooses an integer n large enough such that for any semistable sheaf E with Hilbert polynomial P, the natural evaluation map

$$\varepsilon_n \colon H^0(E(n)) \otimes \mathcal{O}(-n) \to E \qquad (1.1)$$

is surjective and $\dim H^0(E(n)) = P(n)$. Thus, up to the choice of an isomorphism $H^0(E(n)) \cong V$, where V is some fixed $P(n)$-dimensional vector space, we may identify E with a point in the Quot-scheme parametrising quotients of $V \otimes \mathcal{O}(-n)$ with Hilbert polynomial P. Changing the choice of isomorphism is given by the natural action of the reductive group $\mathrm{SL}(V)$ on the Quot-scheme.

For the second step, one chooses another integer $m \gg n$ so that applying the functor $H^0(-\otimes\mathcal{O}(m))$ to (1.1) converts it into a surjective map

$$\alpha_E \colon H^0(E(n)) \otimes H \to H^0(E(m)) \qquad (1.2)$$

where $H = H^0(\mathcal{O}(m-n))$ and $\dim H^0(E(m)) = P(m)$. More precisely, this construction of Grothendieck's is applied after choosing the isomorphism $H^0(E(n)) \cong V$ and thus α_E determines a point in the Grassmannian of $P(m)$-dimensional quotients of $V \otimes H$. The fact that the Quot-scheme is embedded in the Grassmannian means that this point determines the quotient map $V \otimes \mathcal{O}(-n) \to E$. Now $\mathrm{SL}(V)$ acts linearly on the Grassmannian and Simpson's main result [34, Theorem 1.19] is that the GIT-semistable orbits in a certain component of the embedded Quot-scheme are precisely those which correspond to semistable sheaves E.

The essential change of strategy that we make in this paper is to refrain from choosing the isomorphism $H^0(E(n)) \cong V$ and consider

just the combined effect of the two steps above, whereby the sheaf E determines functorially the 'Kronecker module' α_E of (1.2), from which E can in turn be recovered.

A first observation in favour of this change of view-point is that the GIT-semistability of the orbit in the Grassmannian is equivalent to the natural semistability of the Kronecker module α_E (see Remark 2.4). Thus, roughly speaking, the strategy becomes to show that a sheaf E is semistable if and only if the Kronecker module α_E is semistable. However, this is not quite what one is able to prove, which indicates something of the technical difficulty inherent in any formulation of the construction.

Now, a Kronecker module $\alpha \colon V \otimes H \to W$ is precisely the data required to give $V \oplus W$ the structure of a (right) module for the algebra

$$A = \begin{pmatrix} \Bbbk & H \\ 0 & \Bbbk \end{pmatrix}$$

where $\Bbbk$ is the base field. If X is connected and reduced, or more generally if $H^0(\mathcal{O}_X) = \Bbbk$, then $A = \mathrm{End}_X(T)$, where

$$T = \mathcal{O}(-n) \oplus \mathcal{O}(-m).$$

However, the crucial point is that T is always a (left) A-module. Thus, the functor $E \mapsto \alpha_E$ is the natural functor

$$\Phi := \Phi_{n,m} = \mathrm{Hom}_X(T, -) \colon \mathbf{mod}\text{-}\mathcal{O}_X \to \mathbf{mod}\text{-}A, \qquad (1.3)$$

where $\mathbf{mod}\text{-}\mathcal{O}_X$ is the category of coherent sheaves of $\mathcal{O}_X$-modules and $\mathbf{mod}\text{-}A$ is the category of finite dimensional right A-modules; see §2.2 for more discussion.

A second benefit of the functorial approach is that the functor Φ has a left adjoint

$$\Phi^\vee = - \otimes_A T \colon \mathbf{mod}\text{-}A \to \mathbf{mod}\text{-}\mathcal{O}_X \qquad (1.4)$$

which provides an efficient description of how Grothendieck's embedding works in this context. More precisely, the fact that the A-module $\mathrm{Hom}_X(T, E)$ determines the sheaf E amounts to the fact that the natural evaluation map (the 'counit' of the adjunction)

$$\varepsilon_E \colon \mathrm{Hom}_X(T, E) \otimes_A T \to E \qquad (1.5)$$

is an isomorphism. As we shall prove in Theorem 3.4, this holds not just for semistable sheaves, but for all n-regular sheaves, in the sense of Castelnuovo-Mumford (cf. §3.1). Thus we see that Φ induces an 'embedding' of moduli functors

$$f_{n,m} \colon \mathscr{M}_X^{reg}(P) \to \mathscr{M}_A(P(n), P(m))$$

where $\mathscr{M}_X^{reg}$ is the moduli functor of n-regular sheaves, of given Hilbert polynomial, and $\mathscr{M}_A$ is the moduli functor of A-modules, of given dimension vector (cf. §2.2). Note that semistable sheaves are n-regular for large enough n and, indeed, this is one condition imposed on n in

the usual construction. Thus the moduli functor $\mathscr{M}_X^{ss}$ of semistable sheaves embeds in $\mathscr{M}_X^{reg}$ and, in this way, in $\mathscr{M}_A$.

Furthermore, the general machinery of adjunction provides a simple condition for determining when a module M is in the image of the embedding: the adjunction also has a 'unit'

$$\eta_M \colon M \to \operatorname{Hom}_X(T, M \otimes_A T), \qquad (1.6)$$

and $M \cong \operatorname{Hom}_X(T, E)$, for some E for which ε_E is an isomorphism, if and only if η_M is an isomorphism. Hence, we can, in principle, identify which A-modules arise as the image under Φ of n-regular sheaves and, in particular, can prove that the locus of such A-modules is locally closed in any family of modules (Proposition 4.2). This enables us to show that $\mathscr{M}_X^{reg}(P)$ is locally isomorphic to a quotient functor (Theorem 4.5), which provides a key ingredient in our moduli space construction: essentially replacing the Quot-scheme in the usual construction.

In particular, it is now sufficient to show that the functor Φ takes semistable sheaves to semistable A-modules (one part of our main Theorem 5.10), so that we have a locally closed embedding of functors

$$f_{n,m} \colon \mathscr{M}_X^{ss}(P) \to \mathscr{M}_A^{ss}(P(n), P(m))$$

It is known [2, 14] how to construct the moduli space $\mathcal{M}_A^{ss}$ of semistable A-modules in a straightforward manner and we can then use this to construct an *a priori* quasi-projective moduli space $\mathcal{M}_X^{ss}$ of semistable sheaves (Theorem 6.4). Note that, because such moduli spaces actually parametrise S-equivalence classes, it is helpful to observe that the functor Φ respects S-equivalence (another part of Theorem 5.10). We complete the argument using Langton's method to show that $\mathcal{M}_X^{ss}$ is proper and hence projective (Proposition 6.6). Thus we obtain a closed embedding of moduli spaces

$$\varphi_{n,m} \colon \mathcal{M}_X^{ss}(P) \to \mathcal{M}_A^{ss}(P(n), P(m)).$$

As a technical point, note that this embedding is scheme-theoretic in characteristic zero, but in characteristic p we only know that it is scheme-theoretic on the open set of stable points (Proposition 6.7) and set-theoretic at the strictly semistable points.

A significant use of the embedding φ, and the whole functorial approach, is a better understanding of theta functions, i.e. natural homogeneous coordinates on the moduli space. The homogeneous coordinate rings of the moduli spaces $\mathcal{M}_A^{ss}$ are by now well understood through general results about semi-invariants of representations of quivers [4, 30]. More precisely, these rings are spanned by determinantal theta functions of the form $\theta_\gamma(M) = \det \operatorname{Hom}_A(\gamma, M)$ for maps

$$\gamma \colon U_1 \otimes P_1 \to U_0 \otimes P_0,$$

between projective A-modules. In particular, such theta functions detect semi-stability of A-modules (see Theorem 7.1). Now, using the adjunction

$$\text{Hom}_X(\Phi^\vee(\gamma), E) = \text{Hom}_A(\gamma, \Phi(E)),$$

we can write the restriction of such θ_γ to $\mathcal{M}_X^{ss}$ as an explicit theta function $\theta_\delta(E) = \det \text{Hom}_X(\delta, E)$, where

$$\delta = \Phi^\vee(\gamma)\colon U_1 \otimes \mathcal{O}(-m) \to U_0 \otimes \mathcal{O}(-n).$$

Thus, the theta functions θ_δ detect semistability of sheaves (Theorem 7.2) and furthermore, up to the same conditions as on the embedding φ, they can be used to provide a projective embedding of $\mathcal{M}_X^{ss}$ (Theorem 7.10).

This even improves what is known about theta functions on moduli spaces of bundles on smooth curves (Corollary 7.15), because in this case θ_δ coincides with the usual theta function θ_F associated to the bundle $F = \text{coker}\,\delta$.

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2. Background on sheaves and Kronecker modules

In this section, we set out our conventions and review the notions of semistability, stability and S-equivalence for sheaves (in §2.1) and for Kronecker modules (in §2.2). We also explain the equivalence between H-Kronecker modules and right A-modules.

Throughout the paper X is a fixed projective scheme, of finite type over an algebraically closed field $\Bbbk$ of arbitrary characteristic, with a very ample invertible sheaf $\mathcal{O}(1)$. A 'sheaf E on X' will mean a coherent sheaf of $\mathcal{O}_X$-modules.

We use the notation

$$\text{"for } n \gg 0\text{"} \quad \text{to mean} \quad \text{"}\exists n_0 \ \forall n \geq n_0\text{"} \quad \text{and}$$
$$\text{"for } m \gg n \gg 0\text{"} \quad \text{to mean} \quad \text{"}\exists n_0 \ \forall n \geq n_0 \ \exists m_0 \ \forall m \geq m_0\text{"}.$$

Note that this notation does not necessarily imply that $n > 0$ or $m > n$.

2.1. **Sheaves.** Let E be a non-zero sheaf. Its *dimension* is the dimension of the support $\text{Supp}(E) := \{x \in X | E_x \neq 0\} \subset X$. We say that E is *pure* if the dimension of any non-zero subsheaf $E' \subset E$ equals the dimension of E.

The Hilbert polynomial $P(E)$ is given by

$$P(E, \ell) = \chi(E(\ell)) = \sum_{i=0}^{\infty} (-1)^i h^i(E(\ell)),$$

where $h^i(F) = \dim H^i(F)$. It can be shown (e.g. [13, Lemma 1.2.1]) that

$$P(E, \ell) = r\ell^d/d! + \text{ terms of lower degree in } \ell, \qquad (2.1)$$

where d is the dimension of E and $r = r(E)$ is a positive integer, which is roughly the 'rank' of E, or more strictly its 'multiplicity' [13, 18].

Definition 2.1. A sheaf E is *semistable* if E is pure and, for each nonzero subsheaf $E' \subset E$,

$$\frac{P(E')}{r(E')} \leq \frac{P(E)}{r(E)} \tag{2.2}$$

Such an E is *stable* if the inequality (2.2) is strict for all proper E'. The polynomial occurring on either side of (2.2) is called the *reduced Hilbert polynomial* of E' or E.

In this definition, the ordering on polynomials $p, q \in \mathbb{Q}[\ell]$ is lexicographic starting with the highest degree terms. Hence the inequality $p \leq q$ (resp. $p < q$) is equivalent to the condition that $p(n) \leq q(n)$ (resp. $p(n) < q(n)$) for $n \gg 0$.

Any semistable sheaf E has a (not necessarily unique) *S-filtration*, that is, a filtration by subsheaves

$$0 = E_0 \subset E_1 \subset \cdots \subset E_k = E,$$

whose factors E_i/E_{i-1} are all stable with the same reduced Hilbert polynomial as E. The isomorphism class of the direct sum

$$\operatorname{gr} E := \bigoplus_{i=1}^{k} E_i/E_{i-1}$$

is independent of the filtration and two semistable sheaves E and F are called *S-equivalent* if $\operatorname{gr} E \cong \operatorname{gr} F$.

Remark 2.2. As observed by Rudakov [29, §2], the condition of semistability may be formulated in a way that does not *a priori* require purity. For polynomials p, p' with positive leading term, define

$$p' \preccurlyeq p \quad \text{iff} \quad \frac{p'(n)}{p'(m)} \leq \frac{p(n)}{p(m)}, \quad \text{for } m \gg n \gg 0. \tag{2.3}$$

Note: [29] uses a different, but equivalent, formulation in terms of the coefficients of the polynomials.

Then E is semistable if and only if $P(E') \preccurlyeq P(E)$, for all nonzero $E' \subset E$. In particular, a sheaf satisfying this condition is automatically pure, because a polynomial of lower degree is bigger in this ordering.

2.2. **Kronecker modules.** Here and throughout the paper, for integers $m > n$, we consider the sheaf

$$T = \mathcal{O}(-n) \oplus \mathcal{O}(-m), \tag{2.4}$$

together with a finite dimensional $\mathbb{k}$-algebra

$$A = \begin{pmatrix} \mathbb{k} & H \\ 0 & \mathbb{k} \end{pmatrix} \tag{2.5}$$

of operators on T. More precisely, $A = L \oplus H \subset \mathrm{End}_X(T)$, where $L = \Bbbk e_0 \oplus \Bbbk e_1$ is the semisimple algebra generated by the two projection operators onto the summands of T and

$$H = H^0(\mathcal{O}(m - n)) = \mathrm{Hom}(\mathcal{O}(-m), \mathcal{O}(-n)),$$

acting on T in the evident off-diagonal manner. Note that H is an L-bimodule, whose structure is characterised by the equation $e_0 H e_1 = H$. In particular, $H \otimes_L H = 0$ and so A is actually the tensor algebra of H over L.

A right A-module structure on M is thus a right L-module structure together with a right L-module map $M \otimes_L H \to M$. The former is the same as a direct sum decomposition $M = V \oplus W$, where $V = M e_0$ and $W = M e_1$, while the latter is the same as an H-Kronecker module $\alpha \colon V \otimes H \to W$.

Alternatively, we may say that A is the path algebra of the quiver

$$\bullet \xrightarrow{\ H\ } \bullet \tag{2.6}$$

where H is the multiplicity space for the arrow (cf. [9]) or, after choosing a basis for H, indicates that there are $\dim H$ arrows. A representation of this quiver is precisely an H-Kronecker module and the equivalence we have described is the standard one between representations of quivers and modules for their path algebras.

The example of particular interest is $\mathrm{Hom}_X(T, E)$. On one hand, this has a natural right module structure over $A \subset \mathrm{Hom}_X(T, T)$, given by composition of maps. On the other hand, we have the obvious decomposition

$$\mathrm{Hom}_X(T, E) = H^0(E(n)) \oplus H^0(E(m))$$

together with the multiplication map $\alpha_E \colon H^0(E(n)) \otimes H \to H^0(E(m))$, as in (1.2).

Now, the basic discrete invariant of an A-module $M = V \oplus W$ is its *dimension vector* $(\dim V, \dim W)$. A submodule $M' \subset M$ is given by subspaces $V' \subset V$ and $W' \subset W$ such that $\alpha(V' \otimes H) \subset W'$.

Definition 2.3. An A-module $M = V \oplus W$ is *semistable* if, for each nonzero submodule $M' = V' \oplus W'$ of M,

$$\frac{\dim V'}{\dim W'} \leq \frac{\dim V}{\dim W}. \tag{2.7}$$

Such a module is *stable* if the inequality is strict for all proper M'. The ratio occurring on either side of (2.7) is called the *slope* of M' or M. It lies in the (ordered) interval $[0, +\infty]$.

Remark 2.4. If M is semistable and $\dim V > 0$, then we see that $\alpha \colon V \otimes H \to W$ must be surjective, because otherwise $V \oplus \mathrm{im}\,\alpha$ is a destabilising submodule of M. It is also clearly sufficient to impose the inequality (2.7) just for *saturated* submodules, i.e. those for which

$W' = \alpha(V' \otimes H)$. Thus, comparing Definition 2.3 with [34, Proposition 1.14], we see that (isomorphism classes of) semistable/stable Kronecker modules correspond precisely to GIT-semistable orbits in the Grassmannian of $(\dim W)$-dimensional quotients of $V \otimes H$. Of course, Definition 2.3 may be seen directly to be equivalent to a natural GIT-semistability for Kronecker modules; see Theorem 4.8 and Remark 4.9 for further discussion.

As in the case of sheaves, a semistable A-module M admits an S-filtration by submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M,$$

such that the quotients M_i/M_{i-1} are stable with the same slope as M. The isomorphism class of $\operatorname{gr} M := \oplus_{i=1}^k M_i/M_{i-1}$ is independent of the filtration and two semistable A-modules M and N are *S-equivalent* if $\operatorname{gr} M \cong \operatorname{gr} N$.

Remark 2.5. We shall see shortly that, for any fixed sheaf E, the cohomology $H^i(E(n))$, for $i \geq 1$, vanishes for $n \gg 0$, so that the dimension vector of $\operatorname{Hom}_X(T, E)$ is then $(P(E, n), P(E, m))$. Comparing Definition 2.3 with Remark 2.2, we observe that E is a semistable sheaf if and only if, for all nonzero $E' \subset E$, the A-submodule $\operatorname{Hom}_X(T, E')$ does not destabilise $\operatorname{Hom}_X(T, E)$, for $m \gg n \gg 0$.

This provides the basic link between semistability of sheaves and semistability of Kronecker modules, but falls well short of what we need. Our main tasks will be to show that we can choose n, m 'uniformly', i.e. depending only on P_E but not on E or E', and further to show that the submodules $\operatorname{Hom}_X(T, E')$ are the 'essential' ones, i.e. if $\operatorname{Hom}_X(T, E)$ is unstable, then it is destabilised by one of these.

Remark 2.6. Kronecker modules have already played a distinguished role in the study of moduli of sheaves, especially on the projective plane $\mathbb{P}^2$. Barth [1] showed that any stable rank 2 bundle F (with $c_1 = 0$) could be recovered from the Kronecker module

$$\alpha_F \colon H^1(F(-2)) \otimes H \to H^1(F(-1)),$$

where $H = H^0(\mathcal{O}(1))$, and explicitly identified the Kronecker modules that arose in this way. Hulek [12] generalised the analysis to higher rank bundles and made an explicit link between the stability of F and the stability of α_F.

These methods are part of the general 'monad' machinery, which was in due course used by Le Potier [19] to construct all moduli spaces of sheaves on $\mathbb{P}^2$, but using the more general data of Kronecker complexes. A different but related construction, using exceptional bundles, enabled Drezet [2] to show that certain 'extremal' moduli spaces of sheaves on $\mathbb{P}^2$ could be actually identified with certain moduli spaces of Kronecker modules.

3. The embedding functor

In this section, for T and A as in (2.4) and (2.5) and with an additional mild assumption on $m-n$, we show that the functor $\operatorname{Hom}_X(T, -)$ embeds sufficiently nice sheaves in the category of A-modules. Here, 'sufficiently nice' means n-regular, in the sense of Castelnuovo-Mumford (see §3.1), and 'embeds' means that $\operatorname{Hom}_X(T, -)$ is fully faithful. In other words, we prove, in §3.3, that the natural evaluation map

$$\varepsilon_E \colon \operatorname{Hom}_X(T, E) \otimes_A T \to E$$

is an isomorphism, for any n-regular sheaf E. In §3.2, we describe how to construct, for any right A-module M, the sheaf $M \otimes_A T$ and we describe explicitly the adjunction between the functors $\operatorname{Hom}_X(T, -)$ and $- \otimes_A T$. For general background on adjoint functors, see [20].

3.1. Castelnuovo-Mumford regularity.

Definition 3.1 ([26])**.** A sheaf E is n-regular if

$$H^i(E(n - i)) = 0 \quad \text{for all } i > 0. \tag{3.1}$$

We write "regular" for "0-regular".

Because this consists of finitely many open conditions, it follows from Serre's Vanishing Theorem [11, Theorem III.5.2] that any bounded family of sheaves is n-regular for $n \gg 0$. The point of this slightly odd definition is revealed by the following consequences.

Lemma 3.2 ([26] or [13, Lemma 1.7.2])**.** *If E is n-regular, then*
  (1) *E is m-regular for all $m \geq n$,*
  (2) *$H^i(E(n)) = 0$ for all $i > 0$, hence $\dim H^0(E(n)) = P(E, n)$,*
  (3) *$E(n)$ is globally generated, that is, the natural evaluation map $\varepsilon_n \colon H^0(E(n)) \otimes \mathcal{O}(-n) \to E$ is surjective,*
  (4) *the multiplication maps $H^0(E(n)) \otimes H^0(\mathcal{O}(m-n)) \to H^0(E(m))$ are surjective, for all $m \geq n$.*

In particular, by Lemma 3.2(3) there is a short exact sequence

$$0 \to F \longrightarrow V \otimes \mathcal{O}(-n) \xrightarrow{\varepsilon_n} E \to 0, \tag{3.2}$$

where $V = H^0(E(n))$ has dimension $P(E, n)$. If we also know that the 'syzygy' F is m-regular for some $m > n$, then we can obtain a 'presentation' of E of the form

$$U \otimes \mathcal{O}(-m) \xrightarrow{\delta} V \otimes \mathcal{O}(-n) \xrightarrow{\varepsilon_n} E \to 0, \tag{3.3}$$

where $U = H^0(F(m))$ and δ is the composition of the evaluation map $f_m \colon U \otimes \mathcal{O}(-m) \to F$ and the inclusion. Note that $P(E)$ determines the Hilbert polynomial of F and hence the dimension of U.

In fact, as we shall in the proof of Theorem 3.4 in §3.3, the map δ is essentially equivalent to the Kronecker module α_E described in the Introduction and thus we have a procedure to recover E from α_E. This

procedure can also be understood in a more functorial language that
we explain in the next subsection.

First we note that the m-regularity of the syzygy F is actually inde-
pendent of E and requires only that $m \gg n$. More precisely, we have
the following.

Lemma 3.3. *Suppose E is a non-zero n-regular sheaf, F is the syzygy
in (3.2) and $m > n$. Then F is m-regular if and only if $\mathcal{O}(m-n)$ is
regular.*

Proof. The fact that it is necessary for $\mathcal{O}(m-n)$ to be regular follows
immediately by applying the functor H^i to the short exact sequence

$$0 \to F(m-i) \to H^0(E(n)) \otimes \mathcal{O}(m-n-i) \to E(m-i) \to 0,$$

since E is m-regular by Lemma 3.2(1).

On the other hand, to see that the regularity of $\mathcal{O}(m-n)$ is sufficient,
consider the following piece of the same long exact sequence

$$H^{i-1}(E(m-i)) \to H^i(F(m-i)) \to H^0(E(n)) \otimes H^i(\mathcal{O}(m-n-i)).$$

For $i > 1$, the vanishing of $H^i(F(m-i))$ follows from the fact E is
$(m-1)$-regular since $m-1 \geq n$. In the case $i = 1$, we also need that

$$H^0(E(n)) \otimes H^0(\mathcal{O}(m-n-1)) \to H^0(E(m-1))$$

is surjective, which comes from Lemma 3.2(4). $\square$

Since we can certainly choose $m > n$ large enough that $\mathcal{O}(m-n)$ is
regular, this means that every n-regular sheaf E with a fixed Hilbert
polynomial has a presentation by a map δ as in (3.3) with fixed U and
V. Thus, for example, n-regular sheaves with given Hilbert polynomial
are bounded, since the set of such presentations certainly is.

3.2. **The adjoint functor.** To see how to construct $M \otimes_A T$, recall
from §2.2 that the A-module structure on M can be specified by a
direct sum decomposition $M = V \oplus W$, giving the L-module structure,
together with a Kronecker module $\alpha \colon V \otimes H \to W$, or equivalently a
right L-module map $\alpha \colon M \otimes_L H \to M$.

On the other hand, T is a left A-module, with its L-module structure
given by the decomposition $T = \mathcal{O}(-n) \oplus \mathcal{O}(-m)$ and the additional
A-module structure given by the multiplication map

$$\mu \colon H \otimes \mathcal{O}(-m) \to \mathcal{O}(-n), \tag{3.4}$$

which we can also write as a left L-module map $\mu \colon H \otimes_L T \to T$.

From this point of view, $M \otimes_A T$ should be constructed as the quo-
tient of $M \otimes_L T$ by relations expressing the fact that the additional H
action is the same on either side of the tensor product. More precisely,
it is the cokernel of the following map.

$$M \otimes_L H \otimes_L T \xrightarrow{\ 1 \otimes \mu - \alpha \otimes 1\ } M \otimes_L T \tag{3.5}$$

Writing the L-module structure explicitly as a direct sum decomposition gives the following exact sequence.

$$V \otimes H \otimes \mathcal{O}(-m) \xrightarrow{1 \otimes \mu - \alpha \otimes 1} \begin{array}{c} V \otimes \mathcal{O}(-n) \\ \oplus \\ W \otimes \mathcal{O}(-m) \end{array} \xrightarrow{c_n + c_m} M \otimes_A T \to 0$$

(3.6)

The exactness in (3.6) is also equivalent to the fact that the following diagram is a push-out.

$$\begin{array}{ccc} V \otimes \mathcal{O}(-n) & \xrightarrow{c_n} & M \otimes_A T \\ {\scriptstyle 1 \otimes \mu} \uparrow & & \uparrow {\scriptstyle c_m} \\ V \otimes H \otimes \mathcal{O}(-m) & \xrightarrow{\alpha \otimes 1} & W \otimes \mathcal{O}(-m) \end{array}$$

(3.7)

Thus we can see that c_n is surjective if and only if α is surjective.

Note that, when α is surjective, it carries the same information as its kernel $\beta \colon U \to V \otimes H$, which is also equivalent to the map

$$\delta = (1 \otimes \mu) \circ (\beta \otimes 1) \colon U \otimes \mathcal{O}(-m) \to V \otimes \mathcal{O}(-n). \qquad (3.8)$$

In this case, the kernel of c_n is the image of δ, i.e. we have a presentation of $M \otimes_A T$ as the cokernel of δ, as we did for E in (3.3). Indeed, we shall see in the next section that (3.3) is a special case of this construction, when $M = \mathrm{Hom}_X(T, E)$.

Before that, we describe explicitly the adjunction between $- \otimes_A T$ and $\mathrm{Hom}_X(T, -)$, that is, the natural isomorphism between

$$\mathrm{Hom}_X(M \otimes_A T, E) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_X(T, E)).$$

This isomorphism is the first row in the following commutative diagram with exact columns and thus it is induced by the second and third rows.

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathrm{Hom}_X(M \otimes_A T, E) & \xrightarrow{\cong} & \mathrm{Hom}_A(M, \mathrm{Hom}_X(T, E)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_X(M \otimes_L T, E) & \xrightarrow{\cong} & \mathrm{Hom}_L(M, \mathrm{Hom}_X(T, E)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_X(M \otimes_L H \otimes_L T, E) & \xrightarrow{\cong} & \mathrm{Hom}_L(M \otimes_L H, \mathrm{Hom}_X(T, E)) \end{array}$$

(3.9)

Note that the left hand column reflects the construction of $M \otimes_A T$ as the cokernel of (3.5), while the right hand column expresses the fact that an A-module map is an L-module map that commutes with the action of H. The second and third isomorphisms and the commuting of the lower square are clear once you unpack the L-module structure as a direct sum.

For example, the 'unit' of the adjunction $\eta_M \colon M \to \operatorname{Hom}_X(T, E))$ corresponds to $\operatorname{id} \colon E \to E$ when $E = M \otimes_A T$. Thus

$$\eta_M = \eta_n \oplus \eta_m \in \operatorname{Hom}(V, H^0(E(n))) \oplus \operatorname{Hom}(W, H^0(E(m))$$

and is naturally identified with

$$c_n + c_m \in \operatorname{Hom}_X(V \otimes \mathcal{O}(-n), E) \oplus \operatorname{Hom}_X(W \otimes \mathcal{O}(-m), E).$$

On the other hand, the 'counit' $\varepsilon_E \colon M \otimes_A T \to E$, corresponding to the identity when $M = \operatorname{Hom}_X(T, E)$, is induced by the universal property of cokernels from the map $\varepsilon_n \oplus \varepsilon_m \colon M \otimes_L T \to E$ given by the two evaluation maps

$$\varepsilon_n \colon V \otimes \mathcal{O}(-n) \to E \qquad \varepsilon_m \colon W \otimes \mathcal{O}(-m) \to E.$$

Therefore, we also refer to ε_E as the evaluation map.

3.3. **Embedding regular sheaves.** We are now in a position to prove the main result of the section.

Theorem 3.4. *Assume that $\mathcal{O}(m - n)$ is regular. Then the functor $\operatorname{Hom}_X(T, -)$ is fully faithful on the full subcategory of n-regular sheaves. In other words, if E is an n-regular sheaf, then the natural evaluation map $\varepsilon_E \colon \operatorname{Hom}_X(T, E) \otimes_A T \to E$ is an isomorphism.*

Proof. By Lemma 3.3, the assumption means that the syzygy F in (3.2) is m-regular. In particular $H^1(F(m)) = 0$, so applying the functor $H^0(-(m))$ to (3.2) and recalling that $H = H^0(\mathcal{O}(m - n))$, we obtain a short exact sequence

$$0 \longrightarrow U \overset{\beta}{\longrightarrow} V \otimes H \overset{\alpha}{\longrightarrow} W \longrightarrow 0, \qquad (3.10)$$

where $U = H^0(F(m))$, $V = H^0(E(n))$, $W = H^0(E(m))$ and α is the Kronecker module corresponding to the A-module $\operatorname{Hom}_X(T, E)$.

Now (3.2) and (3.10)$\otimes \mathcal{O}(-m)$ form a commutative diagram of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F & \longrightarrow & V \otimes \mathcal{O}(-n) & \overset{\varepsilon_n}{\longrightarrow} & E & \longrightarrow & 0 \\
& & \uparrow{\scriptstyle f_m} & & \uparrow{\scriptstyle 1 \otimes \mu} & & \uparrow{\scriptstyle \varepsilon_m} & & \\
0 & \longrightarrow & U \otimes \mathcal{O}(-m) & \overset{\beta \otimes 1}{\longrightarrow} & V \otimes H \otimes \mathcal{O}(-m) & \overset{\alpha \otimes 1}{\longrightarrow} & W \otimes \mathcal{O}(-m) & \longrightarrow & 0
\end{array}
$$

$$(3.11)$$

where the vertical maps are all natural evaluation maps. As F is m-regular, f_m is surjective and so E is the cokernel of the map

$$\delta = (1 \otimes \mu) \circ (\beta \otimes 1)$$

of (3.3), as already observed after that equation. Alternatively, we see that the following sequence is exact.

$$V \otimes H \otimes \mathcal{O}(-m) \xrightarrow{1 \otimes \mu - \alpha \otimes 1} \begin{array}{c} V \otimes \mathcal{O}(-n) \\ \oplus \\ W \otimes \mathcal{O}(-m) \end{array} \xrightarrow{\varepsilon_n + \varepsilon_m} E \longrightarrow 0 \quad (3.12)$$

Comparing (3.12) with (3.6) we see that $E \cong M \otimes_A T$, where in this case $M = \mathrm{Hom}_X(T, E)$. More precisely, we see that $\varepsilon_E \colon M \otimes_A T \to E$, as described at the end of §3.2, is an isomorphism. $\qquad\square$

Note that, if E is n-regular, then, by Lemma 3.2(1,2), the dimension vector of $\mathrm{Hom}_X(T, E)$ is $(P(E, n), P(E, m))$. Thus all n-regular sheaves with a fixed Hilbert polynomial are embedded in the subcategory of A-modules with a fixed dimension vector, which is a bounded subcategory: a slight variation of the argument at the end of §3.1.

Remark 3.5. In Theorem 3.4, to deduce that the evaluation map ε_E is an isomorphism, we assumed that E is n-regular and the syzygy F in (3.2) is m-regular. In fact, one may readily check that the proof works under slightly weaker hypotheses: either that $E(n)$ is globally generated and F is m-regular, or that E is n-regular and $F(m)$ is globally generated. In the latter case, one should use Lemma 3.2(4) to show that α is surjective.

4. Families and moduli

In this section, we show how the functorial embedding of the previous section determines an embedding of moduli functors $f \colon \mathscr{M}_X^{reg} \to \mathscr{M}_A$ (see §4.3 and §4.4 for definitions). This requires that we first show, in §4.1, that Theorem 3.4 extends to flat families of n-regular sheaves. We then show, in §4.2, how to identify the image of the embedding and show that it is locally closed. We describe, in §4.3, how $\mathscr{M}_A$ is naturally locally isomorphic to a quotient functor of a finite dimensional vector space R by a reductive group G and then use §4.2 to deduce, in §4.4, that $\mathscr{M}_X^{reg}$ is locally isomorphic to a quotient functor of a locally closed subscheme $Q \subset R$ by G.

We conclude this section, in §4.5, by describing how the GIT quotient of R by G is a moduli space $\mathcal{M}_A^{ss}$ for semistable A-modules, corepresenting the functor $\mathscr{M}_A^{ss} \subset \mathscr{M}_A$. This prepares the way for the construction of the moduli space of sheaves in §6, once we have shown that the embedding functor preserves semistability: the main task of §5.

4.1. **Preservation of flat families.** Let S be a scheme. A *flat family E over S of sheaves on X* is a sheaf E on $X \times S$, which is flat over S. On the other hand, a *flat family M over S of right A-modules* is a sheaf M of right modules over the sheaf of algebras $\mathcal{O}_S \otimes A$ on S, which is locally free as a sheaf of $\mathcal{O}_S$-modules.

Let $\pi \colon X \times S \to S$ and $p_X \colon X \times S \to X$ be the canonical projections. The adjoint pair formed by (1.3) and (1.4), extends to an adjoint pair of functors between the category **mod**-$A \otimes \mathcal{O}_S$ of sheaves of right A-modules on S (coherent as $\mathcal{O}_S$-modules) and the category **mod**-$\mathcal{O}_{X \times S}$

of sheaves on $X \times S$,

$$
\begin{array}{c}
\mathbf{mod}\text{-}A \otimes \mathcal{O}_S \\
{\scriptstyle -\otimes_A T} \Big\downarrow\Big\uparrow {\scriptstyle \mathrm{Hom}_X(T,-)} \\
\mathbf{mod}\text{-}\mathcal{O}_{X \times S}
\end{array}
\tag{4.1}
$$

where we are using the abbreviations

$$
\mathrm{Hom}_X(T, E) := \pi_* \mathcal{H}om_{X \times S}(p_X^* T, E),
$$

and

$$
M \otimes_A T := \pi^* M \otimes_{\mathcal{O}_{X \times S} \otimes A} p_X^* T,
$$

for a sheaf E on $X \times S$ and a sheaf of right A-modules M on S.

Proposition 4.1. *Assume $\mathcal{O}(m - n)$ is regular. Let S be any scheme. Then $\mathrm{Hom}_X(T, -)$ is a fully faithful functor from the full subcategory of $\mathbf{mod}\text{-}\mathcal{O}_{X \times S}$ consisting of flat families over S of n-regular sheaves to the full subcategory of $\mathbf{mod}\text{-}A \otimes \mathcal{O}_S$ consisting of flat families over S of A-modules.*

Proof. To deduce that flat families are preserved by the functor, it is sufficient to know that $H^1(E_s(n)) = 0 = H^1(E_s(m))$ for every sheaf E_s in the family, so that $H^0(E_s(n))$ and $H^0(E_s(m))$ have locally constant dimension and hence $\mathrm{Hom}_X(T, E)$ is locally free. This vanishing follows from the regularity of $E_s(n)$, and hence $E_s(m)$, by Lemma 3.2(1).

The result then follows by applying Theorem 3.4 fibrewise, using general results about cohomology and flat base extensions [11]. $\square$

4.2. **The image of the embedding.** One of the uses of the functorial approach of this paper is to identify the image of the n-regular sheaves by the functor $\mathrm{Hom}_X(T, -)$, using its left adjoint $- \otimes_A T$ and the unit of the adjunction. Indeed, a simple general feature of adjunctions (from [20, IV, Theorem 1(ii)]) means that the following two statements are equivalent:

(1) $M \cong \mathrm{Hom}_X(T, E)$ and $\varepsilon_E \colon \mathrm{Hom}_X(T, E) \otimes_A T \to E$ is an isomorphism,
(2) $E \cong M \otimes_A T$ and $\eta_M \colon M \to \mathrm{Hom}_X(T, M \otimes_A T)$ is an isomorphism.

Thus, informally speaking, any module knows how to tell that it is in the image of the embedding and, if so, what sheaf (up to isomorphism) it came from. More precisely, because of Theorem 3.4, a right A-module M is isomorphic to $\mathrm{Hom}_X(T, E)$ for some n-regular sheaf E with Hilbert polynomial P if and only if the unit map η_M is an isomorphism and the sheaf $M \otimes_A T$ is n-regular with Hilbert polynomial P. It is clearly also necessary that M has dimension vector $(P(n), P(m))$.

This simple statement has an important refinement for families of modules, which will play a key part in our subsequent construction of

the moduli of sheaves. Roughly speaking, it says that being in the image of the n-regular sheaves with Hilbert polynomial P is a locally closed condition in any flat family of modules. Note that we use the simplified notation of §4.1 for applying our functors to families.

Proposition 4.2. *Assume $\mathcal{O}(m-n)$ is regular. Let M be a flat family, over a scheme B, of right A-modules of dimension vector $(P(n), P(m))$. There exists a (unique) locally closed subscheme $i\colon B_P^{[reg]} \hookrightarrow B$ with the following properties.*

(a) *$i^*M \otimes_A T$ is a flat family, over $B_P^{[reg]}$, of n-regular sheaves on X with Hilbert polynomial P and the unit map*

$$\eta_{i^*M}\colon i^*M \to \mathrm{Hom}_X(T, i^*M \otimes_A T)$$

is an isomorphism.

(b) *If $\sigma\colon S \to B$ is such that $\sigma^*M \cong \mathrm{Hom}_X(T, E)$ for a flat family E over S of n-regular sheaves on X with Hilbert polynomial P, then σ factors through $i\colon B_P^{[reg]} \hookrightarrow B$ and $E \cong \sigma^*M \otimes_A T$.*

Proof. Consider $F = M \otimes_A T$, which is a sheaf on $X \times B$, but not necessarily flat over B. We can split up B using the flattening stratification for the sheaf F over the projection $X \times B \to B$ (see [26, 13] for details). Thus, there is a locally closed subscheme

$$j\colon B_P \hookrightarrow B, \tag{4.2}$$

whose (closed) points are precisely those $b \in B$ for which the fibres F_b have Hilbert polynomial P, and such that j^*F is a flat family over B_P of sheaves on X with Hilbert polynomial P. Therefore B_P contains an open set C of points b where the sheaf F_b is n-regular and then C contains an open set D of points for which the unit map $\eta_{M_b}\colon M_b \to \mathrm{Hom}_X(T, F_b)$ is an isomorphism. The assertion that $D \subset C$ is open comes from the fact that, restricted to C, F is a flat family of n-regular sheaves and so $\mathrm{Hom}_X(T, F)$ is a flat family of A-modules, by Proposition 4.1. If we set $B_P^{[reg]} = D$, as an open subscheme of B_P, then, by construction, we have a locally closed subscheme $i\colon B_P^{[reg]} \hookrightarrow B$ satisfying (a).

To prove (b), first note that the counit

$$\varepsilon_E\colon \mathrm{Hom}_X(T, E) \otimes_A T \to E$$

is an isomorphism (by Proposition 4.1) and hence the isomorphism $\sigma^*M \cong \mathrm{Hom}_X(T, E)$ implies that $\sigma^*M \otimes_A T \cong E$. In particular, $\sigma^*M \otimes_A T$ is a flat family of n-regular sheaves on X with Hilbert polynomial P. By the universal property of the flattening stratification (see [26, 13]), the fact that $\sigma^*M \otimes_A T$ is a flat family implies that σ factors through $j\colon B_P \hookrightarrow B$, while the fact that this is a family of n-regular sheaves implies that σ factors through $C \subset B_P$. Since the counit map

for E is an isomorphism, the unit map for $\mathrm{Hom}_X(T, E)$ is also an iso-morphism. This, together with the isomorphism $\sigma^* M \cong \mathrm{Hom}_X(T, E)$, imply that the unit map $\eta_{\sigma^* M} \colon \sigma^* M \to \mathrm{Hom}_X(T, \sigma^* M \otimes_A T)$ is an iso-morphism, hence σ factors through $D \subset C$. In other words, σ factors through $i \colon B_P^{[reg]} \hookrightarrow B$, as required. $\qquad\square$

4.3. **Moduli functors of Kronecker modules.** Let H be a finite di-mensional vector space, A the algebra of (2.5) and a, b positive integers. Let V and W be vector spaces of dimensions a and b, respectively. The isomorphism classes of (right) A-modules, i.e. H-Kronecker modules, with dimension vector (a, b) are in natural bijection with the orbits of the *representation space*

$$R := R_A(a, b) = \mathrm{Hom}_{\Bbbk}(V \otimes H, W) \qquad (4.3)$$

by the canonical left action of the symmetry group $\mathrm{GL}(V) \times \mathrm{GL}(W)$, i.e. for $g = (g_0, g_1)$ and $\alpha \in R$,

$$g \cdot \alpha = g_1 \circ \alpha \circ (g_0^{-1} \otimes 1_H).$$

The subgroup $\Delta = \{(t1, t1) | t \in \Bbbk^\times\}$ acts trivially, so we can consider the induced action of the group

$$G = \mathrm{GL}(V) \times \mathrm{GL}(W)/\Delta.$$

Thus, naively, the 'moduli set' parametrising isomorphism classes of A-modules with dimension vector (a, b) is the quotient set R/G of G-orbits in R. Unfortunately, this quotient usually cannot be given the geometric structure of a nice 'space', e.g. a separated scheme.

After Grothendieck, an environment for giving more geometrical sense to such quotients is the category of functors $\mathbf{Sch}^\circ \to \mathbf{Set}$ from schemes to sets. Note first that this category does include the category of schemes itself, because a scheme Z is determined by its functor of points

$$\underline{Z} \colon \mathbf{Sch}^\circ \to \mathbf{Set} \colon X \mapsto \mathrm{Hom}(X, Z).$$

Furthermore, the Yoneda Lemma tells us that every natural trans-formation $\underline{Y} \to \underline{Z}$ is of the form $\underline{f}$ for some morphism of schemes $f \colon Y \to Z$. Note also that $\underline{G}$ is a group valued functor, so we may replace the quotient set R/G by the *quotient functor*

$$\underline{R}/\underline{G} \colon \mathbf{Sch}^\circ \to \mathbf{Set} \colon X \mapsto \underline{R}(X)/\underline{G}(X). \qquad (4.4)$$

In this category of set-valued functors, we also have a replacement for the set of isomorphism classes of A-modules, namely the *moduli functor*

$$\mathscr{M}_A := \mathscr{M}_A(a, b) \colon \mathbf{Sch}^\circ \to \mathbf{Set}, \qquad (4.5)$$

where $\mathscr{M}_A(S)$ is the set of isomorphism classes of families over S of A-modules with dimension vector (a, b), in the sense of §4.1.

Note that, in this context, not all functors are as nice as the functor of points of a scheme, which is a 'sheaf' in an appropriate 'Grothendieck

topology'. Indeed, the moduli functor $\mathscr{M}_A$ and the quotient functor $\underline{R}/\underline{G}$ are not strictly isomorphic, but become so after 'sheafification'.

More concretely, we have the following definition (cf. [34, Section 1]) for the Zariski topology.

Definition 4.3. A natural transformation $g\colon \mathscr{A} \to \mathscr{B}$ between functors $\mathscr{A}, \mathscr{B}\colon \mathbf{Sch}^\circ \to \mathbf{Set}$ is a *local isomorphism* if, for each S in $\mathbf{Sch}$,

   (1) given $a_1, a_2 \in \mathscr{A}(S)$ such that $g_S(a_1) = g_S(a_2)$, there is an open cover $S = \bigcup_i S_i$ such that $a_1|_{S_i} = a_2|_{S_i}$ for all i,
   (2) if $b \in \mathscr{B}(S)$, then there is an open cover $S = \bigcup_i S_i$ and $a_i \in \mathscr{A}(S_i)$ such that $g_{S_i}(a_i) = b|_{S_i}$ for all i.

Since R carries a tautological family $\mathbb{M}$ of A-modules (whose fibre over each $\alpha \in R$ is the A-module defined by the map $\alpha\colon V \otimes H \to W$), we have a natural transformation

$$h\colon \underline{R} \to \mathscr{M}_A \tag{4.6}$$

where h_S assigns to an element of $\underline{R}(S)$, i.e. a map $\sigma\colon S \to R$, the isomorphism class of the pull-back $\sigma^*\mathbb{M}$.

Proposition 4.4. *The natural transformation h induces a local isomorphism $\widetilde{h}\colon \underline{R}/\underline{G} \to \mathscr{M}_A$.*

Proof. First observe that two elements of $\underline{R}(S)$ define isomorphic families of A-modules if and only if they are related by an element of $\underline{G}(S)$. Thus the induced natural transformation is well-defined and satisfies part (1) of Definition 4.3 (even without taking open covers). Part (2) is satisfied, because any family of A-modules over S can be trivialised locally, i.e. there is an open cover $S = \bigcup_i S_i$ such that the restriction to each S_i is the pull-back by a map $S_i \to R$. $\qquad\square$

Note that there are open subfunctors

$$\mathscr{M}_A^s \subset \mathscr{M}_A^{ss} \subset \mathscr{M}_A \tag{4.7}$$

given by requiring that all the modules in the families over S are stable or semistable, respectively. Correspondingly, there are open subsets

$$R^s \subset R^{ss} \subset R \tag{4.8}$$

given by requiring that $\alpha \in R$ is stable or semistable. Proposition 4.4 restricts to give local isomorphisms

$$\underline{R^s}/\underline{G} \to \mathscr{M}_A^s \qquad \text{and} \qquad \underline{R^{ss}}/\underline{G} \to \mathscr{M}_A^{ss} \tag{4.9}$$

which will be provide the route to proving, in Theorem 4.8, that there are moduli spaces of semistable and stable A-modules.

4.4. **Moduli functors of sheaves.** Just as in §4.3, there is a straight-forward moduli functor of sheaves

$$\mathscr{M}_X := \mathscr{M}_X(P) \colon \mathbf{Sch}^\circ \to \mathbf{Set} \tag{4.10}$$

which assigns to each scheme S the set of isomorphism classes of flat families over S of sheaves on X with Hilbert polynomial P. This has open subfunctors

$$\mathscr{M}_X^s \subset \mathscr{M}_X^{ss} \subset \mathscr{M}_X \tag{4.11}$$

given by requiring that all the sheaves in the families are stable or semistable, respectively. There are also open subfunctors

$$\mathscr{M}_X^{reg} := \mathscr{M}_X^{reg}(n) \subset \mathscr{M}_X$$

of n-regular sheaves, for any fixed integer n.

The aim of this subsection is to prove an analogue of Proposition 4.4 for n-regular sheaves.

Theorem 4.5. *For any P and n, the moduli functor $\mathscr{M}_X^{reg}(P,n)$ is locally isomorphic to a quotient functor $\underline{Q}/\underline{G}$.*

Proof. To begin, let $a = P(n)$, $b = P(m)$ and $T = \mathcal{O}(-n) \oplus \mathcal{O}(-m)$, where $m > n$ and $\mathcal{O}(m - n)$ is regular, so that Propositions 4.1 and 4.2 apply. For $A \subset \operatorname{End}_X(T)$, as in (2.5), consider the representation space R as in (4.3) and let $\mathbb{M}$ be the tautological family of A-modules on R.

Let

$$Q = R_P^{[reg]} \subset R$$

be the locally closed subscheme of R satisfying Proposition 4.2 for the flat family $\mathbb{M}$. Roughly, Q parametrises those $\alpha \in R$ that are isomorphic to Kronecker modules arising as $\operatorname{Hom}_X(T, E)$ for E an n-regular sheaf with Hilbert polynomial P. More precisely, the formulation of this in Proposition 4.2 gives the required result, as follows.

Consider the following diagram of functors $\mathbf{Sch}^\circ \to \mathbf{Set}$ and natural transformations between them.

$$\begin{array}{ccc} \underline{Q} & \xrightarrow{\underline{i}} & \underline{R} \\ {\scriptstyle g} \downarrow & & \downarrow {\scriptstyle h} \\ \mathscr{M}_X^{reg} & \xrightarrow{f} & \mathscr{M}_A \end{array} \tag{4.12}$$

Here $\underline{i}$ comes from the inclusion $i \colon Q \hookrightarrow R$, while h is as in (4.6) and, for any scheme S,

$$g_S \colon \underline{Q}(S) \to \mathscr{M}_X^{reg}(S) \colon \sigma \mapsto [\sigma^* \mathbb{M} \otimes_A T],$$
$$f_S \colon \mathscr{M}_X^{reg}(S) \to \mathscr{M}_A(S) \colon [E] \mapsto [\operatorname{Hom}_X(T, E)],$$

where $[\,]$ denotes the isomorphism class.

The diagram (4.12) commutes, because, by Proposition 4.2(a), the unit map $\eta_{\sigma^*\mathbb{M}}$ is an isomorphism for any $\sigma\colon S \to Q$. Thus, there is a natural map

$$\underline{Q}(S) \to \mathscr{M}_X^{reg}(S) \times_{\mathscr{M}_A(S)} \underline{R}(S)\colon \sigma \mapsto (g_S(\sigma), i \circ \sigma)$$

Furthermore, this map is a bijection, by Proposition 4.2(b). In other words, the diagram (4.12) yields a pull-back in **Set** for all S, i.e. it is a pull-back.

Now (4.12) induces another pull-back diagram

$$\begin{array}{ccc} \underline{Q}/\underline{G} & \longrightarrow & \underline{R}/\underline{G} \\ \widetilde{g}\downarrow & & \downarrow\widetilde{h} \\ \mathscr{M}_X^{reg} & \longrightarrow & \mathscr{M}_A \end{array} \qquad (4.13)$$

where $\widetilde{h}$ is the local isomorphism of Proposition 4.4. But the pull-back of a local isomorphism is a local isomorphism, so $\widetilde{g}$ is the local isomorphism we require. $\qquad\square$

4.5. **Moduli spaces.** It is moduli functors, as above, that determine what it means for a scheme to be a moduli space. More precisely, in the terminology introduced by Simpson [34, Section 1], a moduli space is a scheme which 'corepresents' a moduli functor.

Definition 4.6. Let $\mathscr{M}\colon \mathbf{Sch}^\circ \to \mathbf{Set}$ be a functor, $\mathcal{M}$ a scheme and $\psi\colon \mathscr{M} \to \underline{\mathcal{M}}$ a natural transformation. We say that $\mathcal{M}$ (or strictly ψ) *corepresents* $\mathscr{M}$ if for each scheme Y and each natural transformation $h\colon \mathscr{M} \to \underline{Y}$, there exists a unique $\sigma\colon \mathcal{M} \to Y$ such that $h = \underline{\sigma} \circ \psi$:

$$\begin{array}{ccc} & \mathscr{M} & \\ \psi\downarrow & & \searrow^{h} \\ \underline{\mathcal{M}} & \xrightarrow{\underline{\sigma}} & \underline{Y} \end{array}$$

For example, suppose that an algebraic group G acts on a scheme Z. Then a G-invariant morphism $Z \to Y$ is the same as a natural transformation $\underline{Z}/\underline{G} \to \underline{Y}$, where $\underline{Z}/\underline{G}$ is the quotient functor (4.4). Therefore, a G-invariant morphism $Z \to Z/\!/G$ is a 'categorical quotient', in the sense of [25, Definition 0.5], if and only if the natural transformation $\underline{Z}/\underline{G} \to \underline{Z/\!/G}$ corepresents the quotient functor.

The main reason for proving local isomorphism results like Proposition 4.4 and Theorem 4.5 is they may be used in conjunction with the following lemma to show that suitable categorical quotients are moduli spaces.

Lemma 4.7. *If $g\colon \mathscr{A}_1 \to \mathscr{A}_2$ is a local isomorphism and $\psi_1\colon \mathscr{A}_1 \to \underline{Y}$ is a natural transformation, for a scheme Y, then there is a unique*

natural transformation $\psi_2 \colon \mathscr{A}_2 \to \underline{Y}$ *such that* $\psi_1 = \psi_2 \circ g$:

$$
\begin{array}{ccc}
\mathscr{A}_1 & & \\
g \downarrow & \searrow^{\psi_1} & \\
\mathscr{A}_2 & \xrightarrow[\psi_2]{} & \underline{Y}
\end{array}
\qquad (4.14)
$$

Furthermore, ψ_1 *corepresents* $\mathscr{A}_1$ *if and only if* ψ_2 *corepresents* $\mathscr{A}_2$.

Proof. This holds simply because 'locally isomorphic' means 'isomorphic after sheafification' and $\underline{Y}$ is a sheaf (cf. [34, §1, p. 60]). $\qquad \square$

Thus, by Proposition 4.4, the 'moduli space' of all A-modules of dimension vector (a, b) would be the categorical quotient of R by G. Unfortunately, this quotient collapses to just a single point, because every G orbit in R has 0 in its closure, so any G-invariant map on R is constant. In fact, this collapse sensibly corresponds to the fact that all A-modules of a given dimension vector are Jordan-Hölder equivalent, i.e. they have the same simple factors.

To obtain more interesting and useful moduli spaces, e.g. spaces which generically parametrise isomorphism classes of A-modules, one must restrict the class of A-modules that one considers. This is typically why conditions of semistability are introduced, so that Jordan-Hölder equivalence is replaced by S-equivalence, which reduces to isomorphism for a larger class of objects, namely, the stable ones rather than just the simple ones.

Theorem 4.8. *There exist moduli spaces*

$$
\mathcal{M}_A^s(a, b) \subset \mathcal{M}_A^{ss}(a, b)
$$

of stable and semistable A-modules of dimension vector (a, b), where $\mathcal{M}_A^{ss}$ is projective variety, arising as a good quotient $\pi_A \colon R^{ss} \to \mathcal{M}_A^{ss}$, and $\mathcal{M}_A^s$ is an open subset such that the restriction $\pi_A \colon R^s \to \mathcal{M}_A^s$ is a geometric quotient.

Further, the closed points of $\mathcal{M}_A^{ss}$ correspond to the S-equivalence classes of semistable A-modules, and thus the closed points of $\mathcal{M}_A^s$ correspond to the isomorphism classes of stable A-modules.

Proof. This is a special case of a general construction of moduli spaces of representations of quivers [14]. A key step ([14, Proposition 3.1]) is that the open subsets R^{ss} and R^s coincide with the open subsets of semistable and stable points in the sense of Geometric Invariant Theory [25, 27]. Thus, $\mathcal{M}_A^{ss}$ can be defined as the GIT quotient of R by G. In particular, it is a good quotient (see [32, §1] or [27, §3.4]) of R^{ss} by G and it is projective. The general machinery of GIT also ensures that R^s is an open subset of R^{ss} on which the good quotient is geometric, yielding an open subset $\mathcal{M}_A^s \subset \mathcal{M}_A^{ss}$.

Furthermore, by [14, Proposition 3.2], the closed points of $\mathcal{M}_A^{ss}$ correspond to S-equivalence classes of semistable A-modules, which for stable modules are isomorphism classes.

Finally, to see that $\mathcal{M}_A^{ss}$ is a moduli space in the strict sense, we note that a good quotient is, in particular, a categorical quotient and so we may apply Lemma 4.7 to the local isomorphism in (4.9) to obtain the natural transformation ψ_A in the following commutative diagram.

$$
\begin{array}{ccc}
\underline{R^{ss}} & & \\
\Big\downarrow{\scriptstyle h} & \searrow{\scriptstyle \pi_A} & \\
\mathcal{M}_A^{ss} & \xrightarrow{\psi_A} & \underline{\mathcal{M}_A^{ss}}
\end{array}
\tag{4.15}
$$

The argument for $\mathcal{M}_A^s$ is identical. □

Note that, for this special case of Kronecker modules, a slight variant of the construction of moduli spaces was given earlier by Drezet [2], taking a GIT quotient of the projective space $\mathbb{P}(R)$.

Remark 4.9. We claimed in the Introduction that the moduli space $\mathcal{M}_A^{ss}$ is the same as the GIT quotient of the Grassmannian $\mathcal{G}$ of quotients of $V \otimes H$ of dimension $\dim(W)$ by the natural action of $\mathrm{PGL}(V)$. To justify this claim, note that there is a short exact sequence

$$
1 \to \mathrm{GL}(W) \longrightarrow G \longrightarrow \mathrm{PGL}(V) \to 1.
\tag{4.16}
$$

This enables us to make the GIT quotient of R by G in two steps. In the first step, we take the GIT quotient by $\mathrm{GL}(W)$. In this case, the semistable points $R^{sur} \subset R$ are also the stable points and are given precisely by the surjective $\alpha\colon V \otimes H \to W$. Thus we have a free geometric quotient $R^{sur}/\mathrm{GL}(W)$, which is the Grassmannian $\mathcal{G}$ (cf. [23, §8.1]).

The G-equivariant line bundle on R that controls the quotient restricts to a power of $\det W$ and thus yields a $\mathrm{PGL}(V)$-equivariant line bundle on $\mathcal{G}$, which is a power of the $\mathrm{SL}(V)$-equivariant Plücker line bundle $\mathcal{O}_{\mathcal{G}}(1)$. Hence, the second step of taking the usual GIT quotient of $\mathcal{G}$ by $\mathrm{SL}(V)$ will yield the full GIT quotient of R by G, i.e. the moduli space $\mathcal{M}_A^{ss}$, as required.

Thus the GIT-semistable $\mathrm{SL}(V)$ orbits in $\mathcal{G}$ correspond to the GIT-semistable G orbits in R, which correspond to the semistable Kronecker modules, as Simpson effectively showed in [34, Proposition 1.14] by direct calculation (see also Remark 2.4).

5. Preservation of semistability

In this section, we analyse the effect of the functor $\mathrm{Hom}_X(T, -)$ on semistable sheaves. We already know from §3 that the functor is fully faithful on n-regular sheaves, under a mild condition on T, i.e. on n and m. We start, in §5.1, by listing the much stronger conditions on n

and m that we will now require. All conditions have the feature that, having fixed a Hilbert polynomial P, they hold for $m \gg n \gg 0$.

In §5.2, we show that under these conditions the functor $\mathrm{Hom}_X(T, -)$ takes semistable sheaves of Hilbert polynomial P to semistable Kronecker modules and furthermore preserves S-filtrations. In fact we do a little more: Theorem 5.10 shows that amongst n-regular pure sheaves E the semistable ones are precisely those for which $\mathrm{Hom}_X(T, E)$ is a semistable Kronecker module.

In §5.3, we prove, for comparison, a slightly stronger converse, replacing the assumption of n-regularity by the assumption that $E = M \otimes_A T$ for some M. This result is a more direct analogue of the one needed in Simpson's construction of the moduli space and uses slightly stronger conditions than those in §5.1.

5.1. **Sufficient conditions.** For the rest of this paper, we fix a polynomial

$$P(\ell) = r\ell^d/d! + \cdots$$

which is the Hilbert polynomial of some sheaf on X. We start by noting the following variant of the Le Potier-Simpson estimates [18, 34], tailored for our later use.

Theorem 5.1. *There exists an integer N_{LS} such that, for all $n \geq N_{LS}$ and all sheaves E with Hilbert polynomial P*

(a) *if E is pure, then the following conditions are equivalent:*
(1) *E is semistable.*
(2) *$h^0(E(n)) \geq P(n)$ and for each non-zero subsheaf $E' \subset E$,*

$$h^0(E'(n))P \leq P(n)P(E').$$

(b) *if E is semistable and E' is a non-zero subsheaf of E, then*

$$h^0(E'(n))P = P(n)P(E') \quad \Longleftrightarrow \quad P/r = P(E')/r(E').$$

Proof. This follows from [13, Theorem 4.4.1], when the base field has characteristic zero, and [16, Theorem 4.2] for characteristic p.

In fact, for part (a), if E is semistable, then the implication (1)$\Rightarrow$(2) of [13, Theorem 4.4.1] or of [16, Theorem 4.2] gives the inequality

$$h^0(E'(n))r \leq P(n)r(E'). \tag{5.1}$$

In the case when (5.1) is strict, we deduce immediately that

$$h^0(E'(n))P < P(n)P(E')$$

because (5.1) gives the leading term. On the other hand, if we have equality in (5.1), then, by the last part of [13, Theorem 4.4.1] or [16, Theorem 4.2], we have $rP(E') = r(E')P$, and hence, with the equality in (5.1), we get

$$h^0(E'(n))P = P(n)P(E').$$

The converse is immediate, because the leading term of the polynomial inequality is (5.1), so E is semistable by the implication (2)$\Rightarrow$(1) of [13, Theorem 4.4.1] or of [16, Theorem 4.2].

Part (b) follows similarly from the last part of [13, Theorem 4.4.1] or of [16, Theorem 4.2].   □

Remark 5.2. It is natural to ask whether the assumption that E is pure can be dropped from the implication (2)$\Rightarrow$(1) of Theorem 5.1(a), or at least replaced by the assumption that E is n-regular. It is generally possible, when the dimension of X is at least 3, for some impure sheaf E to be n-regular and yet have a subsheaf E' of lower dimension with $H^0(E'(n)) = 0$, so that E' does not violate condition (2). Thus the question is a more delicate one: having fixed $P(E)$, can n be made large enough to avoid this phenomenon? If so, is the condition $n \geq N_{LS}$ already sufficient?

The first conditions are imposed on n.

(C:1) All semistable sheaves with Hilbert polynomial P are n-regular.

(C:2) The Le Potier-Simpson estimates hold, i.e., $n \geq N_{LS}$, for N_{LS} as in Theorem 5.1.

Condition (C:1) can be satisfied because semistable sheaves with a fixed Hilbert polynomial are bounded (see [22, 34, 18] or [13, Theorem 3.3.7] in characteristic zero and [15, §4] in arbitrary characteristic), and any bounded family of sheaves is n-regular for $n \gg 0$. In fact, (C:1) is assumed in the proof of the Le Potier-Simpson estimates [13, Theorem 4.4.1], hence of Theorem 5.1, so (C:2) is essentially stronger than (C:1).

We next impose conditions on m once n has been fixed. The first is familiar from §3:

(C:3) $\mathcal{O}(m - n)$ is regular.

Now, let E be any n-regular sheaf with Hilbert polynomial P and

$$\varepsilon_n \colon V \otimes \mathcal{O}(-n) \to E$$

be the evaluation map, where $V = H^0(E(n))$. Let $V' \subset V$ be any subspace and let E' and F' be the image and kernel of ε_n restricted to $V' \otimes \mathcal{O}(-n)$, i.e. there is a short exact sequence

$$0 \to F' \to V' \otimes \mathcal{O}(-n) \to E' \to 0.$$

Then

(C:4) F' and E' are both m-regular.

(C:5) The polynomial relation

$$P \dim V' \sim P(E') \dim V$$

is equivalent to the numerical relation

$$P(m) \dim V' \sim P(E', m) \dim V,$$

where $\sim$ is one of $>$, $=$ or $<$.

Condition (C:4) can be satisfied because n-regular sheaves E with Hilbert polynomial P form a bounded family and then so do the $V' \subset H^0(E(n))$. Condition (C:5) is finitely many numerical conditions on m because the set of E' occurring in it is bounded and so there are finitely many $P(E')$. Each numerical condition can be satisfied because an inequality between polynomials in $\mathbb{Q}[\ell]$ is equivalent to the same inequality with $\ell = m$, for all sufficiently large values of m.

Note that (C:3) is implied by (C:4) (see Lemma 3.3), but we record it explicitly to make it clear that the results of §3 apply.

5.2. **From sheaves to modules.** Assuming conditions (C:1)-(C:5), we will see how the semistability of a sheaf E with Hilbert polynomial P is related to the semistability of the A-module $\mathrm{Hom}_X(T, E)$.

Our first results explain the role of (C:4) and (C:5). These conditions will guarantee in particular that the 'essential' subsheaves E' of an n-regular sheaf E, namely those with $E'(n)$ globally generated, correspond to the 'essential' submodules of $\mathrm{Hom}_X(T, E)$, namely those which are 'tight' in the sense of the following definition. By "essential" here, we mean those which control semistability (cf. the proofs of Propositions 5.7 and 5.8 below).

Definition 5.3. Let $M' = V' \oplus W'$ and $M'' = V'' \oplus W''$ be submodules of an A-module M. We say that M' is *subordinate* to M'' if

$$V' \subset V'' \qquad \text{and} \qquad W'' \subset W' \qquad (5.2)$$

We say that M' is *tight* if it is subordinate to no submodule other than itself.

Note that if M'' is a submodule and V' and W' are any subspaces satisfying (5.2), then $M' = V' \oplus W'$ is automatically a submodule. Furthermore, every submodule is subordinate to a tight one and a subordinate submodule has smaller or equal slope, which is why the tight submodules are the 'essential' ones.

Lemma 5.4. *Let E be an n-regular sheaf with Hilbert polynomial P and $E' \subset E$ with $E'(n)$ globally generated. Then*

(a) *the evaluation map $\varepsilon_{E'} \colon \mathrm{Hom}_X(T, E') \otimes_A T \to E'$ is an isomorphism,*

(b) *the polynomial relation*

$$h^0(E'(n))P \sim P(n)P(E') \qquad (5.3)$$

is equivalent to the numerical relation

$$h^0(E'(n))P(m) \sim P(n)h^0(E'(m)), \qquad (5.4)$$

where $\sim$ is one of $>$, $=$ or $<$,

(c) $\mathrm{Hom}_X(T, E')$ *is a tight submodule of $\mathrm{Hom}_X(T, E)$.*

Proof. As E is n-regular, $V = H^0(E(n))$ has dimension $P(n)$ and the evaluation map

$$\varepsilon_n \colon V \otimes \mathcal{O}(-n) \to E$$

is surjective. As $E'(n)$ is globally generated, we have a short exact sequence

$$0 \to F' \to V' \otimes \mathcal{O}(-n) \to E' \to 0. \qquad (5.5)$$

where $V' = H^0(E'(n)) \subset V$ and the second map is the restriction of ε_n.

For (a), note that F' are m-regular, by (C:4). Hence, by Remark 3.5, $\varepsilon_{E'}$ is an isomorphism.

For (b), note that E' is m-regular, also by (C:4). Hence (5.4) is equivalent to

$$h^0(E'(n))P(m) \sim P(n)P(E', m)$$

which is equivalent to (5.3) by (C:5).

For (c), let $\alpha \colon V \otimes H \to H^0(E(m))$ be the Kronecker module corresponding to $\mathrm{Hom}_X(T, E)$. Then, the m-regularity of F' implies that $H^0(E'(m)) = \alpha(V' \otimes H)$. Thus $\mathrm{Hom}_X(T, E')$ is tight if there is no proper $V'' \supset V'$ with $\alpha(V'' \otimes H) = H^0(E'(m))$.

Suppose we have such a V'' and let $E'' = \varepsilon_n(V'' \otimes \mathcal{O}(-n))$. As for E', (C:4) implies that E'' is m-regular and $H^0(E''(m)) = \alpha(V'' \otimes H)$. But then $E'' = E'$, as both $E''(m)$ and $E'(m)$ are globally generated, and so

$$V'' \subset H^0(E''(n)) = H^0(E'(n)) = V'.$$

Thus $\mathrm{Hom}_X(T, E')$ is tight. $\qquad \square$

We also have the converse of Lemma 5.4(c).

Lemma 5.5. *Let E be an n-regular sheaf with Hilbert polynomial P and $M = \mathrm{Hom}_X(T, E)$. If $M' \subset M$ is a tight submodule, then $M' = \mathrm{Hom}_X(T, E')$ for some subsheaf of $E' \subset E$. Furthermore, $E'(n)$ is globally generated and $E' \cong M' \otimes_A T$.*

Proof. Let $M' = V' \oplus W'$ be a submodule of $M = V \oplus W$, which corresponds to a Kronecker module $\alpha \colon V \otimes H \to W$. We may define a subsheaf $E' \subset E$ as the image of $V' \otimes \mathcal{O}(-n)$ under the evaluation map $\varepsilon_n \colon V \otimes \mathcal{O}(-n) \to E$. Then $V' \subset H^0(E'(n))$ and $E'(n)$ is globally generated.

We can apply condition (C:4) to the short exact sequence

$$0 \to F' \to V' \otimes \mathcal{O}(-n) \to E' \to 0,$$

to deduce that F' is m-regular, and hence we have a short exact sequence

$$0 \to H^0(F'(m)) \to V' \otimes H \to H^0(E'(m)) \to 0.$$

Thus $H^0(E'(m)) = \alpha(V' \otimes H) \subset W'$ and so M' is subordinate to $\mathrm{Hom}_X(T, E')$. If M' is tight, then $M' = \mathrm{Hom}_X(T, E')$, as required.

Even under the weaker assumption that M' is saturated, i.e. $W' = \alpha(V' \otimes H)$, we can deduce that $E' \cong M' \otimes_A T$, as in the proof of Theorem 3.4 because $F'(m)$ is globally generated. $\qquad\square$

Remark 5.6. In Lemma 5.5 we could even say that $E' = M' \otimes_A T$, meaning that we use the natural isomorphism $M \otimes_A T \to E$ to identify $M' \otimes_A T$ with a subsheaf of E. Note also that, since $- \otimes_A T$ is not generally left exact, it is a non-trivial fact that $M' \otimes_A T \to M \otimes_A T$ is an injection. However, in this case (cf. the end of the proof) this map will actually be injective whenever M' is saturated, although then we would only have $M' \subset \mathrm{Hom}_X(T, E')$, so such an M' would not be recovered from $E' = M' \otimes_A T$. Indeed different saturated M' could give the same E'.

So, as a consequence of Lemma 5.4(c) and Lemma 5.5, we can say that the functors $\mathrm{Hom}_X(T, -)$ and $- \otimes_A T$ provide a one-one correspondence, i.e. mutually inverse bijections, between the subsheaves $E' \subset E$ with $E'(n)$ globally generated and tight submodules $M' \subset M = \mathrm{Hom}_X(T, E)$. Because this correspondence is functorial, it automatically preserves inclusions between subobjects.

We can now compare semistability for E and $\mathrm{Hom}_X(T, E)$.

Proposition 5.7. *Suppose E is n-regular with Hilbert polynomial P. Then $\mathrm{Hom}_X(T, E)$ is semistable if and only if for all $E' \subset E$*

$$h^0(E'(n))P(m) \leq h^0(E'(m))P(n). \tag{5.6}$$

Proof. As E is n-regular, the A-module $M = \mathrm{Hom}_X(T, E)$ has dimension vector $(P(n), P(m))$ and so this inequality is just the condition (2.7) for the submodule $\mathrm{Hom}_X(T, E')$. Thus, if M is semistable, then the inequality must hold. On the other hand to check that M is semistable, it is sufficient to check (2.7) for tight submodules M', which are all of the form $\mathrm{Hom}_X(T, E')$ for some E', by Lemma 5.5. $\qquad\square$

The next two results depend on condition (C:2), i.e. Theorem 5.1, as well as (C:4) and (C:5).

Proposition 5.8. *Suppose E is n-regular and pure, with Hilbert polynomial P. Then E is semistable if and only if, for all $E' \subset E$, the inequality (5.6) holds.*

Proof. Suppose first that E is semistable. To prove (5.6), we start by assuming that $E'(n)$ is globally generated. By Theorem 5.1(a), we have

$$h^0(E'(n))P \leq P(n)P(E'), \tag{5.7}$$

which by Lemma 5.4(b) implies (5.6).

Now for any $E' \subset E$, let $E'_0(n)$ be the subsheaf of $E(n)$ generated by $H^0(E'(n))$. Then, by construction, $E'_0(n)$ is globally generated and $H^0(E'_0(n)) = H^0(E'(n))$. Also $E'_0 \subset E'$, so $h^0(E'_0(m)) \leq h^0(E'(m))$, so (5.6) for E' follows from (5.6) for E'_0.

For the converse, we use the converse implication in Theorem 5.1(a). As E is pure and n-regular, we have $h^0(E(n)) = P(n)$ and we need to prove (5.7) for any $E' \subset E$.

Again, if we assume $E'(n)$ is globally generated, then Lemma 5.4(b) tells us that (5.6) implies (5.7). For arbitrary $E' \subset E$, define E_0' as above. Since we also have $P(E_0') \leq P(E')$, then (5.7) for E' follows from (5.7) for E_0'. □

Proposition 5.9. *Suppose E is semistable with Hilbert polynomial P and $E' \subset E$. Then the following are equivalent:*

(1) *$P(E')/r(E') = P/r$,*
(2) *E' is n-regular and equality holds in (5.6),*
(3) *$E'(n)$ is globally generated and equality holds in (5.6).*

Proof. For (1)⇒(2), note that, since E is semistable and E' has the same reduced Hilbert polynomial, we can see that E' and E/E' are both semistable with the same reduced Hilbert polynomial. Hence $E' \oplus E/E'$ is semistable with Hilbert polynomial P and thus is n-regular, by (C:1). So in particular E' is n-regular. The equality in (5.6) then follows from the equality of reduced Hilbert polynomials.

(2)⇒(3) immediately, because regular sheaves are globally generated.

For (3)⇒(1), note that E is n-regular by (C:1). Since $E'(n)$ is globally generated, we can use Lemma 5.4(b) to deduce that equality in (5.6) implies the polynomial equality

$$h^0(E'(n))P = P(n)P(E'),$$

which implies the required equality of reduced Hilbert polynomials, by Theorem 5.1(b). □

We now combine these propositions to achieve the main aim of this section. Note that we still assume the conditions (C:1)-(C:5).

Theorem 5.10. *Let E be a sheaf on X with Hilbert polynomial P. Then*

(a) *E is semistable if and only if it is n-regular and pure and the A-module $M = \mathrm{Hom}_X(T, E)$ is semistable.*

Furthermore, if E is semistable, then

(b) *E is stable if and only if M is stable,*
(c) *the functors $\mathrm{Hom}_X(T, -)$ and $- \otimes_A T$ provide a one-one correspondence between the subsheaves $E' \subset E$ with the same reduced Hilbert polynomial as E and the submodules $M' \subset M$ with the same slope as M.*
(d) *the functors in (c) preserve factors in this correspondence, i.e. if $E_1 \subset E_2$ correspond to $M_1 \subset M_2$, then*

$$\begin{aligned} M_2/M_1 &\cong \mathrm{Hom}_X(T, E_2/E_1), \\ E_2/E_1 &\cong (M_2/M_1) \otimes_A T. \end{aligned}$$

Proof. Part (a) is the combination of Propositions 5.7 and 5.8. Now suppose that E is semistable. Then (b) is just a special case of (c), when there are no proper subobjects on either side of the correspondence.

To prove (c), first let $E' \subset E$ have the same reduced Hilbert polynomial as E. Then (1)$\Rightarrow$(3) of Proposition 5.9 means that $E'(n)$ is globally generated and $\mathrm{Hom}_X(T, E')$ has the same slope as M.

Conversely, if $M' \subset M$ has the same slope as M, then M' is a tight submodule, because M is semistable, and so by Lemma 5.5, $M' = \mathrm{Hom}_X(T, E')$ with $E' = M' \otimes_A T$ and $E'(n)$ globally generated. But now, by (3)$\Rightarrow$(1) of Proposition 5.9, E' has the same reduced Hilbert polynomial as E.

Hence the one-one correspondence of Remark 5.6 restricts to the one-one correspondence claimed here.

To prove (d), note that the second isomorphism is automatic because $- \otimes_A T$ is right exact. On the other hand, the stronger implication (1)$\Rightarrow$(2) of Proposition 5.9 tells us that the E_i are actually n-regular, so in particular, $\mathrm{Ext}^1_X(T, E_1) = 0$. Hence, when we apply $\mathrm{Hom}_X(T, -)$ to the short exact sequence

$$0 \to E_1 \longrightarrow E_2 \longrightarrow E_2/E_1 \to 0,$$

we obtain a short exact sequence

$$0 \to M_1 \longrightarrow M_2 \longrightarrow \mathrm{Hom}_X(T, E_2/E_1) \to 0,$$

which gives the first isomorphism. □

This theorem has the following immediate consequence.

Corollary 5.11. *Let E be a semistable sheaf on X with Hilbert polynomial P and $M = \mathrm{Hom}_X(T, E)$ be the corresponding semistable A-module. Then*

$$\mathrm{gr}\, M \cong \mathrm{Hom}_X(T, \mathrm{gr}\, E) \tag{5.8}$$

$$\mathrm{gr}\, E \cong (\mathrm{gr}\, M) \otimes_A T. \tag{5.9}$$

Hence a semistable sheaf E' with Hilbert polynomial P is S-equivalent to E if and only if the modules $\mathrm{Hom}_X(T, E)$ and $\mathrm{Hom}_X(T, E')$ are S-equivalent.

Proof. Let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_k = E \tag{5.10}$$

be any S-filtration of E. Applying $\mathrm{Hom}_X(T, -)$ yields a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M, \tag{5.11}$$

whose terms $M_i = \mathrm{Hom}_X(T, E_i)$ all have the same slope as M. Indeed, this is an S-filtration of M, because if we could refine it, then applying $- \otimes_A T$ would give a refinement of (5.10). Thus (5.8) follows, because $M_i/M_{i-1} = \mathrm{Hom}_X(T, E_i/E_{i-1})$.

Now, (5.9) follows by a similar argument, or it follows from (5.8) together with the observation that $\operatorname{gr} E$ is also semistable with Hilbert polynomial P and so the counit

$$\operatorname{Hom}_X(T, \operatorname{gr} E) \otimes_A T \to \operatorname{gr} E$$

is an isomorphism.

The remainder of the corollary is immediate.   □

Thus, $\operatorname{Hom}_X(T, -)$ provides a well-defined and injective set-theoretic map from S-equivalence classes of semistable sheaves, with Hilbert polynomial P, to S-equivalence classes of semistable modules, with dimension vector $(P(n), P(m))$.

Remark 5.12. It is interesting to ask whether the assumption that E is pure is really necessary (for the 'if' implication) in Proposition 5.8 and thus in Theorem 5.10(a). As the need for purity comes in turn from Theorem 5.1(a), this amounts to the question asked in Remark 5.2.

We do not know the answer, but it would perhaps be more interesting if purity is indeed necessary in all these results, indicating that it might be sensible to consider a wider notion of semistability in which certain impure sheaves could be semistable.

5.3. **From modules to sheaves.** We conclude this section with a stronger converse to Theorem 5.10(a), which is included more for completeness and is not essential to the main arguments in the paper. The result and proof follow more closely one of the key steps in Simpson's construction of the moduli space (cf. [34, Theorem 1.19]).

We assume conditions (C:1) and (C:3) of §5.1, but need to modify the other conditions. First, for (C:2), we need an additional part of [13, Theorem 4.4.1(3)], namely that, for all $n \geq N_{LS}$, a sheaf E with Hilbert polynomial P is semistable, provided that it is pure and for each quotient $E \twoheadrightarrow E''$,

$$\frac{P(n)}{r} \leq \frac{h^0(E''(n))}{r(E'')}. \tag{5.12}$$

We also need to strengthen conditions (C:4) and (C:5) by requiring that they apply with ε_n replaced by any surjective map

$$q \colon V \otimes \mathcal{O}(-n) \to E,$$

where V is any $P(n)$-dimensional vector space and E is any sheaf with Hilbert polynomial P. These stronger conditions can be satisfied because, once n is fixed, the sets of maps q and subspaces V' are bounded ([10], [13, Lemma 1.7.6]).

Proposition 5.13. *Let M be a right A-module of dimension vector $(P(n), P(m))$ and E the sheaf $M \otimes_A T$ on X. If M is semistable and E is pure of Hilbert polynomial P, then E is semistable and the unit map $\eta_M \colon M \to \operatorname{Hom}_X(T, E)$ is an isomorphism.*

Proof. Let $M = V \oplus W$, with Kronecker module $\alpha \colon V \otimes H \to W$. As in §3.2, let

$$c_n \colon V \otimes \mathcal{O}(-n) \to E \qquad c_m \colon W \otimes \mathcal{O}(-m) \to E$$

be the canonical maps, which correspond naturally to the two components of η_M

$$\eta_n \colon V \to H^0(E(n)) \qquad \eta_m \colon W \to H^0(E(m)).$$

Since M is semistable, α is surjective and so c_n is surjective (see (3.7)).

Let F be the kernel

$$0 \to F \longrightarrow V \otimes \mathcal{O}(-n) \xrightarrow{c_n} E \to 0.$$

By the stronger version of (C:4) with $V' = V$, both E and F are m-regular. Hence, applying the functor $H^0(-(m))$ to this short exact sequence, we see that $H^0(c_n(m)) \colon V \otimes H \to H^0(E(m))$ is surjective and, since $H^0(c_n(m)) = \eta_m \circ \alpha$, we deduce that η_m is surjective. Also $h^0(E(m)) = P(m) = \dim W$, so η_m is an isomorphism.

Hence, $\ker \eta_M = \ker \eta_n \oplus 0$ and so $\ker \eta_n = 0$, otherwise M would not be semistable. Thus, η_n is injective and to conclude that η_M is an isomorphism, it remains to show that $h^0(E(n)) = P(n) = \dim V$. This will follow once we have shown that E is semistable, and hence n-regular by (C:1).

To prove that E is semistable, we will apply the modified version of (C:2). Let $p \colon E \twoheadrightarrow E''$ be an epimorphism, E' its kernel, and V' and V'' be the kernel and the image of

$$H^0((p \circ c_n)(n)) \colon V \to H^0(E''(n)),$$

respectively. Then $\eta_n(V') \subset H^0(E'(n))$ and the sheaf

$$E_0' = c_n(V' \otimes \mathcal{O}(-n))$$

is a subsheaf of E'. Let $c_n' \colon V' \otimes \mathcal{O}(-n) \to E_0'$ be the restriction of c_n and $M' \subset M$ the submodule given by

$$\alpha' = \eta_m^{-1} \circ H^0(c_n'(m)) \colon V' \otimes H \to W',$$

where $W' = \eta_m^{-1}(H^0(E_0'(m))) \subset W$. As M is semistable,

$$\frac{\dim V'}{\dim W'} \leq \frac{\dim V}{\dim W}.$$

It follows from (C:4) and the fact that η_m is an isomorphism, that

$$\dim W' = h^0(E_0'(m)) = P(E_0', m),$$

so the previous inequality is $P(m) \dim V' \leq P(E_0', m) \dim V$. The stronger version of (C:5) now implies $P \dim V' \leq P(E_0') \dim V$. But $P(E_0') \leq P(E')$, so

$$P \dim V' \leq P(E') \dim V.$$

Since $\dim V = \dim V' + \dim V''$ and $P = P(E') + P(E'')$, this is equivalent to

$$P(E'') \dim V \leq P(E) \dim V''.$$

If E'' has dimension d, then the leading term of this inequality is $r(E'') \dim V \leq r \dim V''$. Since $V'' \subset H^0(E''(n))$ and $\dim V = P(n)$, this implies (5.12), so E is semistable by the modified version of (C:2).

□

6. Moduli spaces of sheaves

In this section, we complete the construction of the moduli space $\mathcal{M}_X^{ss}$ of semistable sheaves, using the formal machinery set up in §4 and the results of §5. In §6.1, we use the moduli space $\mathcal{M}_A^{ss}$ of semistable A-modules, as described in §4.5, to construct the moduli space $\mathcal{M}_X^{ss}$. In characteristic zero, this is simply a locally closed subscheme of $\mathcal{M}_A^{ss}$, but in characteristic p the construction is a little more delicate. At this stage we only know that $\mathcal{M}_X^{ss}$ is a quasi-projective scheme. In §6.2, we use Langton's method to show that $\mathcal{M}_X^{ss}$ is proper and hence projective. In §6.3, we look more closely at what we can say about the embedding of moduli spaces $\varphi \colon \mathcal{M}_X^{ss} \to \mathcal{M}_A^{ss}$, in particular when it is a scheme-theoretic embedding and when only set-theoretic. In §6.4 and §6.5, we discuss some technical enhancements to the construction.

6.1. **Construction of the moduli spaces.** To construct the moduli space of semistable sheaves on X, we start from the analogue of Theorem 4.5 for semistable and stable sheaves.

Proposition 6.1. *For any P, the moduli functors $\mathscr{M}_X^s(P) \subset \mathscr{M}_X^{ss}(P)$ are locally isomorphic to quotient functors $\underline{Q^{[s]}}/\underline{G} \subset \underline{Q^{[ss]}}/\underline{G}$.*

Proof. Choose n large enough that all semistable sheaves of Hilbert polynomial P are n-regular and choose m large enough that $\mathcal{O}(m-n)$ is regular. Define open subsets

$$Q^{[s]} \subset Q^{[ss]} \subset Q,$$

with Q as in Theorem 4.5, to be the open loci where the fibres of the tautological flat family $\mathbb{F} = i^*\mathbb{M} \otimes_A T$ over Q, are stable and semistable, respectively.

Then the restriction of the family $\mathbb{F}$ to $Q^{[ss]}$ determines a natural transformation

$$g^{[ss]} \colon \underline{Q^{[ss]}} \to \mathscr{M}_X^{ss}, \tag{6.1}$$

which is the restriction of the natural transformation $g \colon \underline{Q} \to \mathscr{M}_X^{reg}$ of (4.12) and, by definition of $Q^{[ss]}$, is also the pull-back of g along $\mathscr{M}_X^{ss} \hookrightarrow \mathscr{M}_X^{reg}$. Hence we obtain a local isomorphism

$$\widetilde{g}^{[ss]} \colon \underline{Q^{[ss]}}/\underline{G} \to \mathscr{M}_X^{ss}, \tag{6.2}$$

by restricting the local isomorphism $\widetilde{g} \colon \underline{Q}/\underline{G} \to \mathscr{M}_X^{reg}$ of (4.13).

An identical argument applies to $Q^{[s]}$ and $\mathcal{M}_X^s$.   □

Thus, the main step in constructing the moduli space is to show that $Q^{[ss]}$ has a good quotient and, further, that this restricts to a geometric quotient of $Q^{[s]}$. Before we do this, we note the following (well-known) lemma.

Lemma 6.2. *Let $\pi\colon Z \to Z /\!\!/ G$ be a good quotient for the action of a reductive algebraic group G on a scheme Z and let Y be a G-invariant open subset of Z. Suppose further that, for each G-orbit O in Y, the closed orbit $O' \subset Z$ contained in the orbit closure $\overline{O}$ is also in Y. Then π restricts to a good quotient $Y \to Y /\!\!/ G$, where $Y /\!\!/ G = \pi(Y)$ is an open subset of $Z /\!\!/ G$.*

Proof. For any good quotient, $\pi(Z \setminus Y)$ will be closed in $Z /\!\!/ G$. Since π induces a bijection between the closed orbits in Z and the (closed) points in $Z /\!\!/ G$ and $\pi(O) = \pi(O')$, the additional assumption here implies that $\pi(Y)$ and $\pi(Z \setminus Y)$ are disjoint. As π is surjective,

$$\pi(Y) = (Z /\!\!/ G) \setminus \pi(Z \setminus Y)$$

and so $\pi(Y)$ is open in $Z /\!\!/ G$. Furthermore, $Y = \pi^{-1}(\pi(Y))$ and so the restriction $\pi\colon Y \to \pi(Y)$ is a good quotient, because this is a property which is local in $Z /\!\!/ G$ (cf. [32, Definition 1.5]).   □

To apply this lemma, we now suppose that n, m satisfy the conditions (C:1)-(C:5) of §5.1, so $Q^{[ss]}$ is locally closed in R^{ss}, by Theorem 5.10.

Proposition 6.3. *The good quotient $\pi_A\colon R^{ss} \to \mathcal{M}_A^{ss}$ of Theorem 4.8 and the inclusion $i\colon Q^{[ss]} \to R^{ss}$ determine a (unique) commutative diagram*

$$\begin{array}{ccc} Q^{[ss]} & \overset{i}{\longrightarrow} & R^{ss} \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_A} \\ \mathcal{M}_X^{ss} & \overset{\varphi}{\longrightarrow} & \mathcal{M}_A^{ss} \end{array} \qquad (6.3)$$

where $\mathcal{M}_X^{ss}$ is quasi-projective, π_X is a good quotient and φ induces a set-theoretic injection on closed points. In characteristic zero, φ is the inclusion of the locally closed subscheme $\pi_A\big(Q^{[ss]}\big)$.

Proof. Let $Y = Q^{[ss]}$ and $\overline{Y}$ be the closure of Y in R. Let $Z = \overline{Y} \cap R^{ss}$, which is closed in R^{ss}. Observe that the additional assumption of Lemma 6.2 holds in this case. The closed orbit in the orbit closure of a point in $Q^{[ss]}$, corresponding to a module $M = \operatorname{Hom}_X(T, E)$, is the orbit corresponding to the associated graded module $\operatorname{gr} M$ (by [14, Proposition 3.2]). But $\operatorname{gr} M \cong \operatorname{Hom}_X(T, \operatorname{gr} E)$, by Corollary 5.11, and $\operatorname{gr} E$ is semistable, so this closed orbit is also in $Q^{[ss]}$.

Thus, to obtain the quasi-projective good quotient $\mathcal{M}_X^{ss} = Y /\!\!/ G$ with the given additional properties, it is sufficient, using Lemma 6.2,

to show that the closed subscheme $Z \subset R^{ss}$ has a projective good quotient with corresponding additional properties.

In characteristic zero, the processes of taking quotient rings and invariant subrings commute (as seen by using the Reynolds operator), and so the (scheme-theoretic) image $\pi_A(Z)$ is the good quotient $Z /\!\!/ G$. Furthermore, $\pi_A(Z)$ is a closed subscheme of $\mathcal{M}_A^{ss}$, which is projective, and hence $Z /\!\!/ G$ is projective.

Thus we have a commutative diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{\ \bar{i}\ } & R^{ss} \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi_A} \\
Z /\!\!/ G & \xrightarrow{\ \overline{\varphi}\ } & \mathcal{M}_A^{ss}
\end{array}
\qquad (6.4)
$$

where the horizontal maps $\bar{i}$ and $\overline{\varphi}$ are closed scheme-theoretic embeddings.

In characteristic p, there is no Reynolds operator and the two processes above do not commute. Hence we cannot construct the good quotient $Z /\!\!/ G$ as the image $\pi_A(Z)$. In this case, we recall that $\mathcal{M}_A^{ss}$ is a GIT quotient of R and R^{ss} is the set of GIT semistable points ([14]). Therefore the GIT quotient of the affine scheme $\overline{Y}$ is a projective scheme (for the same reason as $\mathcal{M}_A^{ss}$ is [14]) and this GIT quotient is the good quotient of $\overline{Y} \cap R^{ss} = Z$. We can then (uniquely) complete the diagram (6.4), because the vertical maps π and π_A are categorical quotients. We further deduce that $\overline{\varphi}$ is closed set-theoretic embedding because $\bar{i}$ is a closed (scheme-theoretic) embedding and the two quotients are good. $\qquad\square$

We are now in a position to complete the promised 'functorial construction' of the moduli space of semistable sheaves.

Theorem 6.4. *The scheme $\mathcal{M}_X^{ss}(P)$, constructed in Proposition 6.3, is the moduli space of semistable sheaves on X of Hilbert polynomial P, i.e. it corepresents the moduli functor $\mathscr{M}_X^{ss}(P)$. The closed points of $\mathcal{M}_X^{ss}$ correspond to the S-equivalence classes of semistable sheaves. Furthermore, there is an open subscheme $\mathcal{M}_X^{s} \subset \mathcal{M}_X^{ss}$ which corepresents the moduli functor $\mathscr{M}_X^{s}$ of stable sheaves and whose closed points correspond to the isomorphism classes of stable sheaves.*

Proof. To see that $\mathcal{M}_X^{ss}$ corepresents $\mathscr{M}_X^{ss}$, we apply Lemma 4.7 to the local isomorphism $\widetilde{g}^{[ss]}\colon \underline{Q^{[ss]}/G} \to \mathscr{M}_X^{ss}$ of (6.1) to obtain the natural transformation ψ_X in the following commutative diagram

$$
\begin{array}{ccc}
& \underline{Q^{[ss]}} & \\
{\scriptstyle g^{[ss]}}\big\downarrow & \!\!\!\!\!\!\searrow{\scriptstyle \pi_X} & \\
\underline{\mathscr{M}_X^{ss}} & \xrightarrow{\ \psi_X\ } & \underline{\mathcal{M}_X^{ss}}
\end{array}
\qquad (6.5)
$$

and use the fact that π_X, from (6.3), is a good, hence categorical, quotient.

Now, the morphism $\varphi \colon \mathcal{M}_X^{ss} \to \mathcal{M}_A^{ss}$ of (6.3) induces a bijection between the closed points of $\mathcal{M}_X^{ss}$ and the closed points of

$$\pi_A\left(Q^{[ss]}\right) \subset \mathcal{M}_A^{ss}.$$

Hence Theorem 4.8 implies that the closed points of $\mathcal{M}_X^{ss}$ correspond to the S-equivalence classes of semistable A-modules that are of the form $\mathrm{Hom}_X(T, E)$ for semistable sheaves E of Hilbert polynomial P. However, we also know from Corollary 5.11 that $\mathrm{Hom}_X(T, E)$ and $\mathrm{Hom}_X(T, E')$ are S-equivalent A-modules if and only if E and E' are S-equivalent sheaves. Thus the closed points of $\mathcal{M}_X^{ss}$ correspond to the S-equivalence classes of semistable sheaves.

For the parts of the theorem concerning stable sheaves, recall from Theorem 5.10(b) that a semistable sheaf E is stable if and only if the A-module $\mathrm{Hom}_X(T, E)$ is stable. Hence $Q^{[s]} = Q^{[ss]} \cap R^s$, where R^s is the open set of stable points. In particular, all G-orbits in $Q^{[s]}$ are closed in $Q^{[ss]}$, because they are closed in R^{ss}. Therefore, we may apply Lemma 6.2 to deduce that $Q^{[s]}$ has a good (in fact, geometric) quotient $\mathcal{M}_X^s = Q^{[s]}/\!\!/ G = \pi_X\left(Q^{[s]}\right)$ which is open in $\mathcal{M}_X^{ss} = Q^{[ss]}/\!\!/ G$. As above, $\mathcal{M}_X^s$ corepresents $\mathscr{M}_X^s$, by Lemma 4.7.

Finally, the closed points of $\mathcal{M}_X^s$ correspond to the isomorphism classes of stable sheaves because $\mathcal{M}_X^s = \pi_X\left(Q^{[s]}\right)$ and 'S-equivalence' for stable sheaves is 'isomorphism'.   □

The functorial nature of the construction can be summarised in the following commutative diagram of natural transformations,

$$\begin{array}{ccc}
\mathscr{M}_X^{ss} & \xrightarrow{\ f\ } & \mathscr{M}_A^{ss} \\
{\scriptstyle \psi_X}\downarrow & & \downarrow{\scriptstyle \psi_A} \\
\underline{\mathcal{M}_X^{ss}} & \xrightarrow{\ \varphi\ } & \underline{\mathcal{M}_A^{ss}}
\end{array} \qquad (6.6)$$

where f is induced by the functor $\mathrm{Hom}_X(T, -)$, as in (4.12), and ψ_A is the corepresenting transformation of (4.15). If $\mathcal{M}_X^{ss}$ and ψ_X are to exist, with the required properties that ψ_X corepresents $\mathscr{M}_X^{ss}$ and induces a bijection between S-equivalence classes and points of $\mathcal{M}_X^{ss}$, then the map φ must exist and be an injection on points. On the other hand, the logic of the construction is that we can effectively show that ψ_X does exist by constructing $\mathcal{M}_X^{ss}$ and φ as in (6.3).

6.2. **Properness of the moduli.** We already know from Proposition 6.3 that $\mathcal{M}_X^{ss}$ is quasi-projective, hence to show that it is projective, it is sufficient to show that it is proper. The basic tool for this is the method of Langton [17] (see also [21, §5]).

Theorem 6.5. *Let C be the spectrum of a discrete valuation ring and C_0 the generic point. If F is a flat family over C_0 of semistable sheaves on X, then F extends to a flat family of semistable sheaves over C.*

Proof. First (cf. the proof of [13, Theorem 2.2.4]) note that F extends to a flat family over C, which can then be modified at the closed point, by [13, Theorem 2.B.1], to obtain a flat family of semistable sheaves. $\square$

Using this we prove the following.

Proposition 6.6. *The moduli space $\mathcal{M}_X^{ss}$ is proper and hence projective.*

Proof. We use the valuative criterion for properness. Let $\Delta = \operatorname{Spec} D$ and $\Delta_0 = \operatorname{Spec} K$, where D is a discrete valuation ring with field of fractions K. Given any $x_0 \colon \Delta_0 \to \mathcal{M}_X^{ss}$, we need to show that x_0 extends to a map $x \colon \Delta \to \mathcal{M}_X^{ss}$, i.e. $x_0 = x \circ j$, where $j \colon \Delta_0 \hookrightarrow \Delta$ is the inclusion.

The first step is to lift $x_0 \in \underline{\mathcal{M}_X^{ss}}(\Delta_0)$ along the natural transformation $\psi_X \colon \mathscr{M}_X^{ss} \to \underline{\mathcal{M}_X^{ss}}$ of (6.5) to obtain a family to which we can apply Langton's method.

In fact, this lift can only be achieved up to a finite cover, i.e. we must take a finite field extension $K' \supset K$, with corresponding covering map $p_0 \colon \Delta_0' \to \Delta_0$, in order to find $y_0 \in \underline{Q^{[ss]}}(\Delta_0')$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\Delta_0' & \xrightarrow{\ y_0\ } & Q^{[ss]} \\
{\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle \pi_X} \\
\Delta_0 & \xrightarrow{\ x_0\ } & \mathcal{M}_X^{ss}
\end{array}
$$

Then the 'lift' of x_0 is $[E_0] = g_{\Delta_0'}^{[ss]}(y_0) \in \mathscr{M}_X^{ss}(\Delta_0')$, since this family has classifying map

$$
(\psi_X)_{\Delta_0'}([E_0]) = \left(\underline{\pi_X}\right)_{\Delta_0'}(y_0) = \pi_X \circ y_0 = x_0 \circ p_0. \tag{6.7}
$$

In other words, $E_0 = y_0^* \mathbb{F}$, where $\mathbb{F} = i^* \mathbb{M} \otimes_A T$ is the tautological family of semistable sheaves on $Q^{[ss]}$.
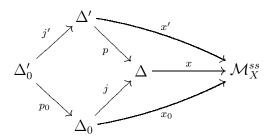
Now, let D' be a discrete valuation ring dominating D, with field of fractions K' Let $p \colon \Delta' \to \Delta$ be the corresponding covering map and $j' \colon \Delta_0' \hookrightarrow \Delta'$ be the inclusion. By Theorem 6.5, E_0 extends over Δ' to a flat family E of semistable sheaves on X, i.e. $E_0 = j'^* E$. But, E has classifying map

$$
x' = (\psi_X)_{\Delta'}([E]) \colon \Delta' \to \mathcal{M}_X^{ss}
$$

and, by the naturality of ψ_X,

$$
x_0 \circ p_0 = (\psi_X)_{\Delta_0'}(j'^* E) = j'^*(\psi_X)_{\Delta'}(E) = x' \circ j'.
$$

Since D and D' are discrete valuation rings and D' dominates D, it follows that $D = K \cap D'$. In other words, the diamond below is a push-out and therefore there exists a map $x\colon \Delta \to \mathcal{M}_X^{ss}$ making the whole diagram commute.

In particular, $x_0 = x \circ j$, as required. $\qquad\qquad\qquad\qquad\square$

6.3. **Conclusions about the embedding of moduli spaces.** In this subsection, we look more closely at the 'parameter space' $Q^{[ss]} \subset R$ of semistable sheaves embedded by the functor $\mathrm{Hom}_X(T, -)$, and the induced embedding of $\mathcal{M}_X^{ss}$ in $\mathcal{M}_A^{ss}$.

Note that, for the purposes of our construction, the most significant facts were that $Q^{[ss]}$ is locally closed and that $Q^{[ss]} \subset R^{ss}$, which is the "only if" part of Theorem 5.10(a). Using the "if" part we can naturally say more.

Indeed, recall that $Q^{[ss]}$ was defined as the open subset of semistable sheaves inside the parameter space Q of embedded n-regular sheaves and note that this is not necessarily the same as $Q \cap R^{ss}$. However, if we define Q^{pur} as the open subset pure sheaves inside Q, then Theorem 5.10(a) does in fact tell us that

$$Q^{[ss]} = Q^{pur} \cap R^{ss}. \qquad\qquad (6.8)$$

In §5.3 we proved a stronger result, with stronger assumptions on n, m. If R_P^{pur} is the subset of R_P (cf. (4.2)) consisting of pure sheaves of Hilbert polynomial P (not necessarily embedded), then Proposition 5.13 says that

$$Q^{[ss]} = R_P^{pur} \cap R^{ss}. \qquad\qquad (6.9)$$

Simpson's construction uses an even stronger result [34, Theorem 1.19], which, in the light of Remark 4.9, is effectively that

$$Q^{[ss]} = \overline{R_P^{pur}} \cap R^{ss}, \qquad\qquad (6.10)$$

where $\overline{R_P^{pur}}$ is the closure in R and hence is an affine scheme. This is what enables Simpson to construct $\mathcal{M}_X^{ss} = Q^{[ss]}/\!\!/G$ as the *a priori* projective GIT quotient of $\overline{R_P^{pur}}$.

By contrast, in this paper, we used Langton's method to discover *a posteriori* that $Q^{[ss]}/\!\!/G$ is proper. In the notation of Proposition 6.3, i.e. $Y = Q^{[ss]}$ and $Z = \overline{Y} \cap R^{ss}$, we may then deduce that $Y/\!\!/G$ is closed in the (separated) GIT quotient $Z/\!\!/G$. Therefore, as $Y/\!\!/G$ is

also dense, it is equal to $Z /\!/ G$. Since $Y = \pi^{-1}(\pi(Y))$ (cf. Lemma 6.2), this means that $Y = Z$, i.e.

$$Q^{[ss]} = \overline{Q^{[ss]}} \cap R^{ss}. \tag{6.11}$$

Thus, we now could say, with weaker assumptions on n, m than Simpson needs, that $\mathcal{M}_X^{ss}$ is the GIT quotient of the affine scheme $\overline{Q^{[ss]}} \subset R$.

In conclusion, we have

Proposition 6.7. *The map* $\varphi \colon \mathcal{M}_X^{ss}(P) \to \mathcal{M}_A^{ss}(P(n), P(m))$ *in* (6.3) *is a closed set-theoretic embedding of projective schemes. This embedding is scheme-theoretic in characteristic zero, while in characteristic p it is scheme-theoretic on the stable locus.*

Furthermore, the GIT construction yields an ample line bundle L on $\mathcal{M}_A^{ss}$ such that $\varphi^* L$ is an ample line bundle on $\mathcal{M}_X^{ss}$.

Proof. First recall that $\mathcal{M}_A^{ss}$ is projective by construction, while $\mathcal{M}_X^{ss}$ is projective by Proposition 6.6. From above, (6.11) means that φ coincides with the map $\overline{\varphi}$ in (6.4) and hence is a closed embedding (scheme-theoretic in characteristic zero, but only set-theoretic in characteristic p). To see that φ is a scheme-theoretic embedding on the stable locus even in characteristic p, consider the restriction of the diagram (6.3) to the stable loci.

$$
\begin{array}{ccc}
Q^{[s]} & \xrightarrow{\ i\ } & R^s \\
{\scriptstyle \pi_X}\big\downarrow & & \big\downarrow{\scriptstyle \pi_A} \\
\mathcal{M}_X^s & \xrightarrow{\ \varphi\ } & \mathcal{M}_A^s
\end{array}
\tag{6.12}
$$

In (6.3) we know that i is a closed (scheme-theoretic) embedding by (6.11). Hence, in (6.12) we also know that i is a closed embedding, because $Q^{[s]} = Q^{[ss]} \cap R^s$ by Theorem 5.10(b). On the other hand, we know that a stable module (or a stable sheaf) is 'simple', in the sense that its endomorphism algebra is just $\Bbbk$. Thus G acts freely on R^s and so π_A is a principal G-bundle over $\mathcal{M}_A^s$. But a closed G-invariant subscheme of a principal bundle is a principal bundle over a closed subscheme of the base and thus the restricted φ in (6.12) is a scheme-theoretic closed embedding.

Finally, note that $\mathcal{M}_A^{ss}$ is constructed as a GIT quotient of R, with respect to a G-linearised (trivial) line bundle $\mathcal{L}$, for which some power $\mathcal{L}^N$ descends from R^{ss} to an ample line L on $\mathcal{M}_A^{ss}$. On the other hand, we now know that $\mathcal{M}_X^{ss}$ can be constructed as a GIT quotient of $\overline{Q^{[ss]}}$ with respect to the restriction of $\mathcal{L}$ and, by the diagram (6.3), the restriction of $\mathcal{L}^N$ to $Q^{[ss]}$ descends to $\varphi^* L$, which is therefore ample. $\square$

In Section 7 we will look more closely at the line bundle $\mathcal{L}$ (and its G-invariant sections) and we will see that, in fact, all powers of $\mathcal{L}$ descend.

One could also deduce that $\varphi^* L$ is ample directly from the ampleness of L, because φ is a finite map (cf. [11, Ch III, Ex 5.7]).

6.4. Uniform/universal properties of moduli spaces.

The moduli spaces of stable and semistable sheaves actually enjoy a stronger property than corepresenting the moduli functors.

Definition 6.8. Let $\mathscr{M} \colon \mathbf{Sch}^\circ \to \mathbf{Set}$ be a functor, $\mathcal{M}$ a scheme and $\psi \colon \mathscr{M} \to \underline{\mathcal{M}}$ a natural transformation. We say that $\mathcal{M}$ *universally corepresents* $\mathscr{M}$ if for each morphism of schemes $\mathcal{N} \to \mathcal{M}$, the fibre product functor $\underline{\mathcal{N}} \times_{\underline{\mathcal{M}}} \mathscr{M}$ is corepresented by the canonical projection $\underline{\mathcal{N}} \times_{\underline{\mathcal{M}}} \mathscr{M} \to \underline{\mathcal{N}}$. If this holds only for *flat* morphisms $\mathcal{N} \to \mathcal{M}$, then we say that $\mathcal{M}$ *uniformly corepresents* $\mathscr{M}$.

This enhancement of Definition 4.6 is a direct generalisation of a similar enhancement of 'categorical quotient' in [25, Definition 0.7]. Indeed, if an algebraic group G acts on a scheme Z, then a G-invariant morphism $Z \to Z /\!/ G$ is a uniform/universal categorical quotient if and only if the induced natural transformation $\underline{Z/G} \to \underline{Z /\!/ G}$ uniformly/universally corepresents $\underline{Z/G}$. In the situation of Lemma 4.7, we can conclude that ψ_1 uniformly/universally corepresents $\mathscr{A}_1$ if and only if ψ_2 uniformly/universally corepresents $\mathscr{A}_2$.

We can enhance Lemma 6.2 by adding "uniform" (or "universal" in characteristic zero) to "good quotient", because

$$Y = \pi^{-1}(Y /\!/ G) = Y /\!/ G \times_{Z /\!/ G} Z.$$

We can similarly enhance Proposition 6.3. In the characteristic zero case, this is because π_A is a universal good quotient, because it is a GIT quotient. This property is then automatically inherited by π, because the diagram (6.4) is a pull-back. In the characteristic p case, we must be more direct: π is a uniform good quotient, because it is a GIT quotient.

Thus we can enhance Theorem 6.4 by adding "uniformly" (or "universally" in characteristic zero) to "corepresents".

6.5. Relative moduli spaces.

The functorial method also enables us to construct relative moduli spaces for a projective morphism of schemes $\rho \colon X \to Y$, with a relatively very ample invertible sheaf $\mathcal{O}(1)$.

If $\mathbf{Sch}/Y$ is the category of schemes over Y, then we must work with the relative moduli functors

$$\mathscr{M}_X^{ss}, \mathscr{M}_X^s \colon (\mathbf{Sch}/Y)^\circ \to \mathbf{Set}$$

defined as in §6.1, where now a 'flat family over S of sheaves on X' is an S-flat sheaf on $X \times_Y S$.

Let $T = \mathcal{O}(-n) \oplus \mathcal{O}(-m)$ and, for $H = \rho_*(\mathcal{O}(m-n))$, let

$$A = \begin{pmatrix} \mathcal{O}_Y & H \\ 0 & \mathcal{O}_Y \end{pmatrix},$$

an $\mathcal{O}_Y$-algebra which naturally acts on T. In other words, there is an $\mathcal{O}_Y$-algebra morphism $A \to \rho_* \mathcal{E}nd_X(T)$, defined using the structure map $\mathcal{O}_Y \to \rho_* \mathcal{O}_X$, the projection operators and the obvious H-action. Thus, we have a functor from sheaves of $\mathcal{O}_X$-modules on X to sheaves of A-modules on Y, written simply $\mathrm{Hom}_X(T, -) := \rho_* \mathcal{H}om_X(T, -)$. This functor also has a left adjoint, written $- \otimes_A T$.

Now a sheaf E on X is *n-regular* (relative to ρ) if $R^i \rho_*(E(n-i)) = 0$ for all $i > 0$. Then (cf. [7, Ch V, Prop 2.2]) there is a relative version of Lemma 3.2 and hence relative versions of Lemma 3.3 and Theorem 3.4.

An additional technical point in the relative case, is that we should (and can) choose $m-n$ large enough that any base change of $\mathcal{O}(m-n)$ is regular and further the formation of $\rho_*(\mathcal{O}(m-n))$ commutes with base change (see [34, Lemma 1.9] and the remark following it). Under this additional assumption, there are relative versions of Proposition 4.1 and hence Proposition 4.2, where a flat family of A-modules over a relative scheme $\sigma \colon S \to Y$ is a locally free sheaf of right modules over $\sigma^*(A)$. The relative version of condition (C:3) in §5.1 should be enhanced by adding this assumption.

Now we have relative moduli functors

$$\mathscr{M}_A^{ss} \subset \mathscr{M}_A \colon (\mathbf{Sch}\,/Y)^\circ \to \mathbf{Set}$$

defined as in §4.3, but using flat families of A-modules over relative schemes. To construct a relative moduli space of Kronecker modules (as in Theorem 4.8), choose free (or even locally free) sheaves V and W over Y of ranks $P(n)$ and $P(m)$, respectively. There is a scheme R over Y, which parametrises representations of A on $V \oplus W$, or equivalently morphisms $H \to \mathcal{H}om_Y(V, W)$. In other words, R represents the functor $(\mathbf{Sch}\,/Y)^\circ \to \mathbf{Set}$ which assigns to any $\sigma \colon S \to Y$ the set $\mathrm{Hom}_S(\sigma^* H, \sigma^* \mathcal{H}om_Y(V, W))$. Note that the existence of this scheme depends on the fact that $\mathcal{H}om_Y(V, W)$ is locally free [28, Theorem 3.5].

The group (scheme over Y) $G = \mathrm{GL}(V) \times \mathrm{GL}(W)/\Delta$ acts naturally on R. If $R^{ss} \subset R$ is the open set of points corresponding to semistable modules over the fibres of A, then the quotient functors $\underline{R^{ss}/G} \subset \underline{R/G}$ are locally isomorphic (over Y) to $\mathscr{M}_A^{ss} \subset \mathscr{M}_A$, as in §4.3. Now GIT constructs a scheme $\mathcal{M}_A^{ss}$, projective over Y, and a (relative) good quotient $R^{ss} \to \mathcal{M}_A^{ss}$ for the G-action. Thus $\mathcal{M}_A^{ss}$ corepresents $\mathscr{M}_A^{ss}$.

Applying the relative version of Proposition 4.2 to the tautological family $\mathbb{M}$ of right A-modules on R, we obtain a Y-scheme $Q = R_P^{[reg]} \subset R$, which satisfies a relative version of Theorem 4.5. Choosing n large enough that condition (C:1) of §5.1 is satisfied for the fibre of ρ over every point of Y, we can define $Q^{[s]} \subset Q^{[ss]} \subset Q$ and obtain a relative version of Proposition 6.1. Choosing n and m large enough that conditions (C:2), (C:4) and (C:5) of §5.1 are also satisfied for the fibre of ρ over every point of Y, Theorem 5.10 and Corollary 5.11 hold fibre-wise. As the formation of $\rho_*(\mathcal{O}(m-n))$ commutes with base change,

the fibre of H over a point of Y is the space of sections of $\mathcal{O}(m - n)$ restricted to the corresponding fibre of ρ. Thus, Theorem 5.10 and Corollary 5.11 apply to the fibres of $\operatorname{Hom}_X(T, \mathbb{F})$ over the points of Y, with $\mathbb{F}$ as in Proposition 6.1. Then we see that $Q^{[ss]}$ is locally closed in R^{ss} and we construct $\mathcal{M}_X^{ss}$ as a relative good quotient $Q^{[ss]} /\!/ G$, as in Proposition 6.3.

Hence we obtain a relative version of Theorem 6.4 in which $\mathcal{M}_X^{ss}$ is a scheme over Y corepresenting the relative moduli functor $\mathscr{M}_X^{ss}$ and the closed points of $\mathcal{M}_X^{ss}$ correspond to the S-equivalence classes of semistable sheaves over the fibres of ρ. There is a similar modification for $\mathcal{M}_X^s$. In addition, the proof of Proposition 6.6 applies in the relative case, so $\mathcal{M}_X^{ss}$ is proper, and hence projective, over Y.

Finally, note that this enhancement can be combined with the enhancement in §6.4 by adding "uniformly" (or "universally" in characteristic zero) to "corepresents".

7. Determinantal theta functions

In this section, we interpret the main results of the paper in terms of determinantal theta functions on the moduli space $\mathcal{M}_X^{ss}$ of sheaves, using analogous results already proved by Schofield & Van den Bergh [30] and Derksen & Weyman [4] for the moduli space $\mathcal{M}_A^{ss}$ of Kronecker modules, or more generally, for moduli spaces of representations of quivers. A key ingredient is the adjunction that has been central to this paper, between $\Phi = \operatorname{Hom}_X(T, -)$ and $\Phi^\vee = - \otimes_A T$.

In §7.1, we give a new characterisation of semistable sheaves, amongst n-regular pure sheaves E, as those which invert certain maps of vector bundles. The condition is implicitly equivalent to the semistability of $\operatorname{Hom}_X(T, E)$, but without any explicit reference to Kronecker modules. This means that semistable sheaves are characterised by the nonvanishing of the corresponding determinantal theta functions, which we describe in more formal detail in §7.2, showing in particular how to interpret them as sections of line bundles on the moduli space. Using stronger results of [30, 4], we show in §7.3, that determinantal theta functions can actually be used to give a projective embedding the moduli space $\mathcal{M}_X^{ss}$, modulo the technical problems with semistable points in characteristic p that we have already encountered. Finally, in §7.4, we explain how the results of §7.3 improve what was known even in the case when X is a smooth curve.

7.1. **A determinantal characterisation of semistability.** To see how semistable A-modules can be characterised, note that, as a right A-module $A = P_0 \oplus P_1$, where $P_0 = e_0 A$ and $P_1 = e_1 A$ are the indecomposable projective modules. If M is an A-module, then the corresponding Kronecker module $\alpha \colon V \otimes H \to W$ is given by the composition map,

after noting that

$$V = \mathrm{Hom}_A(P_0, M), \quad W = \mathrm{Hom}_A(P_1, M), \quad H = \mathrm{Hom}_A(P_1, P_0).$$

Now, a corollary of the results of [30, 4] can be formulated as saying that semistable A-modules are precisely those which invert certain maps between projective modules.

Theorem 7.1. *An A-module $M = V \oplus W$ is semistable if and only if there is a map*

$$\gamma \colon U_1 \otimes P_1 \to U_0 \otimes P_0, \tag{7.1}$$

where U_i are (non-zero) vector spaces, such that the induced linear map

$$\mathrm{Hom}_A(\gamma, M) \colon \mathrm{Hom}(U_0, V) \to \mathrm{Hom}(U_1, W)$$

is invertible, i.e. $\theta_\gamma(M) := \det \mathrm{Hom}_A(\gamma, M) \neq 0$.

Proof. First note that, the fact that $\mathrm{Hom}_A(\gamma, M)$ may be invertible requires in particular that

$$\dim U_0 \dim V = \dim U_1 \dim W. \tag{7.2}$$

When this holds, θ_γ is a G-equivariant polynomial function on the representation space R of (4.3), with values in the one-dimensional G-vector space

$$(\det V)^{-\dim U_0} \otimes (\det W)^{\dim U_1}.$$

Identifying this space with $\Bbbk$, we can consider θ_γ as a semi-invariant with weight

$$\chi_U \colon G \to \Bbbk^\times \colon (g_0, g_1) \mapsto (\det g_0)^{-\dim U_0} (\det g_1)^{\dim U_1}, \tag{7.3}$$

that is, $\theta_\gamma(g \cdot \alpha) = \chi_U(g)\theta_\gamma(\alpha)$ for all $\alpha \in R$ and $g \in G$. By [14], GIT-semistable points $\alpha \in R$ with respect to the character χ_U correspond one-one with semistable A-modules M, because the condition from [14] on submodules $M' = V' \oplus W'$ that

$$\dim U_1 \dim W' - \dim U_0 \dim V' \geq 0,$$

is equivalent to the condition (2.7) by (7.2). In other words, M (or α) is semistable if and only if there is some semi-invariant θ with $\theta(\alpha) \neq 0$, where θ has weight χ_U for some U_0, U_1 satisfying (7.2). But, by [30, Theorem 2.3] or [4, Theorem 1], the space of semi-invariants of weight χ_U is spanned by the 'determinantal semi-invariants' of the form θ_γ and so the result follows.

More precisely, in the notation of [30, §3], we have $\theta_\gamma = P_\phi$, where $\gamma = \hat{\phi}$, and, in the notation of [4], we have $\theta_\gamma = c^N$, where γ is a projective resolution of N.

It is an extra observation, in this case, that γ may be chosen with the particular domain and codomain of (7.1). From the perspective of

[4], this occurs because N must also be semistable and hence, in particular, saturated. From the perspective of [30], it is an elementary computation that inverting a general map between projective A-modules is equivalent to inverting a map of this form. $\qquad\square$

From this theorem for A-modules, we may derive a similar determinantal characterisation of semistability for sheaves on X.

Theorem 7.2. *For a fixed polynomial P, suppose that n, m satisfy the conditions (C:1)-(C:5) of §5.1.*

Let E be an n-regular pure sheaf of Hilbert polynomial P. Then E is semistable if and only if there is a map

$$\delta : U_1 \otimes \mathcal{O}(-m) \to U_0 \otimes \mathcal{O}(-n) \tag{7.4}$$

where U_i are (non-zero) vector spaces, such that the induced linear map

$$\mathrm{Hom}_X(\delta, E) : \ \mathrm{Hom}(U_0, H^0(E(n))) \to \mathrm{Hom}(U_1, H^0(E(m)))$$

is invertible, i.e. $\theta_\delta(E) := \det \mathrm{Hom}_X(\delta, E) \neq 0$.

Proof. As $\Phi^\vee(A) = T$, so $\Phi^\vee(P_0) = \mathcal{O}(-n)$ and $\Phi^\vee(P_1) = \mathcal{O}(-m)$. Thus $\Phi^\vee$ gives a bijection between the maps δ in (7.4) and the maps γ in (7.1). Furthermore, the adjunction between Φ and $\Phi^\vee$ implies that

$$\mathrm{Hom}_X(\Phi^\vee(\gamma), E) = \mathrm{Hom}_A(\gamma, \Phi(E)). \tag{7.5}$$

Thus, the existence of δ with $\theta_\delta(E) \neq 0$ is equivalent to the semistability of $\Phi(E)$, by Theorem 7.1, which is equivalent to the semistability of E, by Theorem 5.10(a). $\qquad\square$

Remark 7.3. The "if" part of Theorem 7.2 has a more direct and elementary proof. Note first that, if such a δ exists, then, as the domain and codomain of $\mathrm{Hom}_X(\delta, E)$ must certainly have the same dimension, we know that

$$P(n) \dim U_0 = P(m) \dim U_1, \tag{7.6}$$

because E is n-regular. Now, if E were not semistable, then by Proposition 5.8, there would exist a subsheaf $E' \subset E$ with

$$h^0(E'(n))P(m) > P(n)h^0(E'(m))$$

and thus

$$h^0(E'(n)) \dim U_0 > h^0(E'(m)) \dim U_1.$$

In other words, if we write $K_0 = U_0 \otimes \mathcal{O}(-n)$ and $K_1 = U_1 \otimes \mathcal{O}(-m)$, then

$$\dim \mathrm{Hom}_X(K_0, E') > \dim \mathrm{Hom}_X(K_1, E').$$

Hence the map

$$\mathrm{Hom}_X(\delta, E') : \ \mathrm{Hom}_X(K_0, E') \to \mathrm{Hom}_X(K_1, E')$$

has non-zero kernel, i.e. there is a non-zero map $\phi : K_1 \to E'$, with $\phi \circ \delta = 0$. But, as $E' \subset E$, this also shows that the kernel of $\mathrm{Hom}_X(\delta, E)$ is non-zero, contradicting the assumption. Thus E must be semistable.

Remark 7.4. One may interpret Theorem 7.2 in terms of derived categories, using the derived adjunction

$$\mathbf{R}\operatorname{Hom}_X(\mathbf{L}\Phi^\vee(N), E) \cong \mathbf{R}\operatorname{Hom}_A(N, \mathbf{R}\Phi(E)). \qquad (7.7)$$

As E is n-regular, $\mathbf{R}\Phi(E) = \Phi(E)$, while, by [4, Theorem 1] and [30, Corollary 1.1], the semistability of $\Phi(E)$ is equivalent to the existence of an A-module N, which is 'perpendicular' to $\Phi(E)$ in the sense that the right-hand side of (7.7) vanishes. On the other hand, the complex $K_\bullet$ that represents $\mathbf{L}\Phi^\vee(N)$ is obtained by taking a projective resolution of N given by a map γ as in (7.1) and applying $\Phi^\vee$. In other words, we obtain the map δ as in (7.4), interpreted as a 2-step complex

$$K_\bullet = K_1 \xrightarrow{\delta} K_0.$$

Observe that, again as E is n-regular, the perpendicularity condition

$$\mathbf{R}\operatorname{Hom}_X(K_\bullet, E) = 0$$

is precisely the condition $\theta_\delta(E) \neq 0$ of Theorem 7.2.

7.2. Theta functions on moduli spaces.

We now explain how the 'functions' θ_γ and θ_δ of §7.1 can be properly interpreted as sections of line bundles on the moduli spaces $\mathcal{M}_A^{ss}$ and $\mathcal{M}_X^{ss}$ with certain universal properties, describing first in some detail the case of A-modules.

Consider a flat family $M = V \oplus W$ over S of A-modules with dimension vector (a, b). Then for any $\gamma \colon U_1 \otimes P_1 \to U_0 \otimes P_0$, with

$$a \dim U_0 = b \dim U_1, \qquad (7.8)$$

we may define a line bundle over S

$$\lambda_U(M) := (\det \operatorname{Hom}(U_0, V))^{-1} \otimes \det \operatorname{Hom}(U_1, W)$$

and a global section

$$\theta_\gamma(M) := \det \operatorname{Hom}_A(\gamma, M) \in H^0(S, \lambda_U(M)).$$

Roughly, we have a natural assignment, to each module M, of a one-dimensional vector space together with a vector in it. More formally, in the sense of [3, §3.1], we have a line bundle λ_U, together with a global section θ_γ, on the moduli functor $\mathscr{M}_A(a, b)$ of (4.5). This means that, given another family M' over S' such that $M' \cong \sigma^*M$ for $\sigma \colon S' \to S$, there is an isomorphism $\sigma^*\lambda_U(M) \cong \lambda_U(M')$ which identifies $\sigma^*\theta_\gamma(M)$ with $\theta_\gamma(M')$. Furthermore, these identifications are functorial in σ.

What we can show is that the restriction of this formal line bundle and section to the moduli functor $\mathscr{M}_A^{ss}$ of (4.7) descends to a genuine line bundle and section on the moduli space $\mathcal{M}_A^{ss}$, in the following sense.

Proposition 7.5. *There is a unique line bundle $\lambda_U(a, b)$ on the moduli space $\mathcal{M}_A^{ss}(a, b)$ and a global section $\theta_\gamma(a, b)$ such that, for any family M over S of semistable A-modules of dimension vector (a, b), we have*

$$\lambda_U(M) \cong \psi_M^*\lambda_U(a, b), \qquad \theta_\gamma(M) = \psi_M^*\theta_\gamma(a, b),$$

where $\psi_M := (\psi_A)_S([M]) \colon S \to \mathcal{M}_A^{ss}(a,b)$ is the classifying map coming from (4.15).

Proof. Let $\mathbb{M}$ be the tautological family of A-modules on $R^{ss}(a,b)$, as in §4.3. Then (7.8) implies that $\lambda_U(\mathbb{M})$ is a G-linearised line bundle. Thus, using Kempf's descent condition [13, Theorem 4.2.15], we see that $\lambda_U(\mathbb{M})$ descends to a (unique) line bundle $\lambda_U(a,b)$ on the good quotient $R^{ss}(a,b)/\!\!/G = \mathcal{M}_A^{ss}(a,b)$ if and only if for each point $\alpha \in R^{ss}$ in a closed orbit, the isotropy group of α acts trivially on the fibre over α. Note that, as $\pi_A = \psi_{\mathbb{M}}$, if it exists, then $\lambda_U(a,b)$ must be the descent of $\lambda_U(\mathbb{M})$.

By [14, Proposition 3.2], a point α of R^{ss} is in a closed orbit if and only if the module $M = \mathbb{M}_\alpha$ is 'polystable', that is,

$$M \cong \bigoplus_{i=1}^k K_i \otimes M_i,$$

where M_i are non-isomorphic stable modules of dimension vector (a_i, b_i) with the same slope as M and K_i are multiplicity vector spaces. Since stable modules are simple, the isotropy group of α for the action of $\mathrm{GL}(V) \times \mathrm{GL}(W)$ is isomorphic to

$$\mathrm{Aut}\, M \cong \prod_{i=1}^k \mathrm{GL}(K_i).$$

By standard properties for determinants of sums and tensor products,

$$\lambda_U(M) \cong \bigotimes_{i=1}^k \lambda_U(K_i \otimes M_i) \cong \bigotimes_{i=1}^k (\det K_i)^{\nu_i} \otimes \lambda_U(M_i)^{\dim K_i},$$

as linear representations of the isotropy group, where

$$\nu_i := b_i \dim U_1 - a_i \dim U_0 = 0,$$

by (7.8), because $a_i/b_i = a/b$. Thus, the isotropy group acts trivially, as required.

Furthermore, $\theta_\gamma(\mathbb{M})$ is a G-invariant section of $\lambda_U(\mathbb{M})$ and so it descends to a section $\theta_\gamma(a,b)$ of $\lambda_U(a,b)$, because the descent means that $\lambda_U(a,b)$ is the G-invariant push-forward of $\lambda_U(\mathbb{M})$.

The universal properties of $\lambda_U(a,b)$ and $\theta_\gamma(a,b)$ follow now from a careful analysis of the local isomorphism $\underline{R^{ss}}/\underline{G} \to \mathcal{M}_A^{ss}$ in (4.9) along similar lines to [3, §3.2]. $\square$

Note that the notation λ_U emphasizes the fact that the line bundle depends on the pair of vector spaces U_0, U_1, but not on the map γ. Note also that

$$\lambda_{U \oplus U'} = \lambda_U \otimes \lambda_{U'}, \qquad \theta_{\gamma \oplus \gamma'} = \theta_\gamma \theta_{\gamma'}. \tag{7.9}$$

This applies equally to the line bundles $\lambda_U(M)$ on families and the line bundles $\lambda_U(a,b)$ on the moduli space $\mathcal{M}_A^{ss}(a,b)$, because the later

descend from a specific case of the former and pull-back commutes with tensor product.

Indeed, up to isomorphism λ_U depends only on $\dim U_0$ and $\dim U_1$ and so, with the constraint (7.8), all possible λ_U are isomorphic to positive powers of a single λ_U with $\dim U_0$ and $\dim U_1$ coprime.

Proposition 7.6. *The line bundle $\lambda_U(a, b)$ on $\mathcal{M}_A^{ss}(a, b)$ is ample. Furthermore, its space of global sections is canonically isomorphic to the space of semi-invariants on R with the weight χ_U of (7.3).*

Proof. Let $\mathbb{M}$ be the tautological family of A-modules on the whole of R. As shown in the proof of Theorem 7.1, $\lambda_U(\mathbb{M})$ is the G-linearised line bundle used in [14] to construct $\mathcal{M}_A^{ss}$ as a GIT quotient, i.e. $\lambda_U(\mathbb{M})$ is the line bundle $\mathcal{L}$ in the proof of Proposition 6.7. Thus, as the restriction of $\lambda_U(\mathbb{M})$ to R^{ss} does descend to the quotient, by Proposition 7.5, the descended line bundle $\lambda_U(a, b)$ is ample and can be taken to be the line bundle L in the statement of Proposition 6.7.

As also shown in the proof of Theorem 7.1, the semi-invariants on R with the weight χ_U are identified with the G-invariant sections of $\lambda_U(\mathbb{M})$. As R is normal (cf. [32, Theorem 4.1(ii)]), we have canonical isomorphisms

$$H^0\left(R, \lambda_U(\mathbb{M})\right)^G = H^0\left(R^{ss}, \lambda_U(\mathbb{M})\right)^G = H^0\left(\mathcal{M}_A^{ss}(a, b), \lambda_U(a, b)\right).$$

$\square$

Note that this last canonical isomorphism identifies the sections $\theta_\gamma(a, b)$ of Proposition 7.5 with the corresponding determinantal semi-invariants $\theta_\gamma(\mathbb{M})$.

Most of the above carries over similarly to the case of sheaves. Given a map $\delta \colon U_1 \otimes \mathcal{O}(-m) \to U_0 \otimes \mathcal{O}(-n)$, where the vector spaces U_0, U_1 satisfy

$$P(n) \dim U_0 = P(m) \dim U_1, \tag{7.10}$$

we obtain a line bundle λ_U with a section θ_δ on the moduli functor $\mathcal{M}_X^{reg}(P)$ of n-regular sheaves with Hilbert polynomial P. This is defined in the analogous way, i.e. given a family E over S of n-regular sheaves with Hilbert polynomial P, we have a line bundle over S

$$\lambda_U(E) := (\det \operatorname{Hom}_X(U_0, E(n)))^{-1} \otimes \det \operatorname{Hom}_X(U_1, E(m))$$

and a global section

$$\theta_\delta(E) := \det \operatorname{Hom}_X(\delta, E) \in H^0(S, \lambda_U(E)),$$

with the appropriate functorial properties.

Note that every such δ is of the form $\Phi^\vee(\gamma)$, for $\gamma = \Phi(\delta)$ and hence the adjunction between Φ and $\Phi^\vee$ yields the identification

$$\operatorname{Hom}_X(\delta, E) = \operatorname{Hom}_A(\gamma, M),$$

for $M = \Phi(E)$. Hence we naturally have

$$\lambda_U(E) \cong \lambda_U(M), \qquad \theta_\delta(E) = \theta_\gamma(M). \qquad (7.11)$$

This essentially tells us how the theta functions restrict under the embedding of Proposition 6.7,

$$\varphi \colon \mathcal{M}_X^{ss}(P) \to \mathcal{M}_A^{ss}(a, b),$$

where $(a, b) = (P(n), P(m))$ and n, m satisfy the conditions (C:1)-(C:5) of §5.1. More precisely, we can use Propositions 7.5 and 7.6 to obtain an analogous result for $\mathcal{M}_X^{ss}$.

Proposition 7.7. *There is an ample line bundle $\lambda_U(P) = \varphi^*\lambda_U(a, b)$ on the moduli space $\mathcal{M}_X^{ss}(P)$ and a global section $\theta_\delta(P) = \varphi^*\theta_\gamma(a, b)$, for $\gamma = \Phi(\delta)$, such that, for any family E over S of semistable sheaves with Hilbert polynomial P, we have*

$$\lambda_U(E) \cong \psi_E^*\lambda_U(P), \qquad \theta_\delta(E) = \psi_E^*\theta_\delta(P),$$

where $\psi_E := (\psi_X)_S([E]) \colon S \to \mathcal{M}_X^{ss}(P)$ is the classifying map coming from (6.5).

Proof. Firstly, $\lambda_U(P)$ is ample, because $\lambda_U(a, b)$ is the ample line bundle L of Proposition 6.7: see the proof of Proposition 7.6.

Recall further that $\mathcal{M}_X^{ss}(P)$ was constructed as a good quotient of a subscheme $Q^{[ss]} \subset R^{ss}$ as in (6.3) which carries a tautological family $\mathbb{F}$ of semistable sheaves, with $\Phi(\mathbb{F}) \cong i^*\mathbb{M}$. Using (7.11) and (6.3), we see that

$$\lambda_U(\mathbb{F}) = i^*\lambda_U(\mathbb{M}) = \pi_X^*\varphi^*\lambda_U(a, b)$$
$$\theta_\delta(\mathbb{F}) = i^*\theta_\gamma(\mathbb{M}) = \pi_X^*\varphi^*\theta_\gamma(a, b)$$

that is, $\lambda_U(\mathbb{F})$ and $\theta_\delta(\mathbb{F})$ descend to $\lambda_U(P)$ and $\theta_\delta(P)$ as defined in the proposition. The universal properties now follow, as in the proof of Proposition 7.5, or by direct argument from (6.6). $\square$

7.3. **The separation property.** We now use the full force of the results of [4, 30] to obtain stronger results about the determinantal theta functions of sheaves θ_δ. The point is that the determinantal theta functions of modules θ_γ don't just detect semistable A-modules, they actually span the ring of semi-invariants on R, which means in particular that they furnish a full set of homogeneous coordinates on the moduli space $\mathcal{M}_A^{ss}$.

Theorem 7.8. *For any dimension vector (a, b), we can find vector spaces U_0, U_1 satisfying (7.8) and finitely many maps*

$$\gamma_0, \ldots, \gamma_N \colon U_1 \otimes P_1 \to U_0 \otimes P_0$$

such that the map

$$\Theta_\gamma \colon \mathcal{M}_A^{ss}(a, b) \to \mathbb{P}^N \colon [M] \mapsto (\theta_{\gamma_0}(M) : \cdots : \theta_{\gamma_N}(M)) \qquad (7.12)$$

is a scheme-theoretic closed embedding.

Proof. In (7.12), one should interpret $\theta_{\gamma_i}(M)$ as the value at $[M]$ of the section $\theta_{\gamma_i}(a, b)$ of the ample line bundle $\lambda_U(a, b)$ on $\mathcal{M}_A^{ss}(a, b)$. Thus we are simply describing a morphism to $\mathbb{P}^N$ given by the linear system spanned by $N + 1$ sections of a line bundle.

We use the identification in Proposition 7.6 of sections of $\lambda_U(a, b)$ with semi-invariants on R, together with the fact from [4, 30] that such semi-invariants are spanned by determinantal ones, to deduce that we can always choose a basis of sections of $\lambda_U(a, b)$ of the form $\theta_{\gamma_i}(a, b)$.

Hence the result follows by choosing U_0, U_1 so that $\lambda_U(a, b)$ is very ample. This is possible by the observation preceding Proposition 7.6.

Alternatively, the result can be proved in a more generic, but less controlled way, i.e. without the results of §7.2. The construction of $\mathcal{M}_A^{ss}$ as a GIT quotient of the representation space R means that it may be written as $\mathrm{Proj}(\mathcal{S})$, where $\mathcal{S}$ is the ring of semi-invariant functions on R. Thus, for some large k, there is a projective embedding determined by the linear system $\mathcal{S}_k$, which has a basis of determinantal semi-invariants, by [4, 30], giving the required result. $\square$

Remark 7.9. By the universal property described in Proposition 7.5, one may also interpret the morphism $\Theta_\gamma \colon \mathcal{M}_A^{ss}(a, b) \to \mathbb{P}^N$ of (7.12) in a more functorial way as the unique morphism associated to a natural transformation of functors $\Theta_\gamma^\natural \colon \mathscr{M}_A^{ss}(a, b) \to \underline{\mathbb{P}^N}$ by the fact that $\mathcal{M}_A^{ss}$ corepresents $\mathscr{M}_A^{ss}$ (cf. Definition 4.6).

The natural transformation $\Theta_\gamma^\natural$ is defined by essentially the same formula as (7.12), i.e. it takes $[M]$ in $\mathscr{M}_A^{ss}(S)$ to the element of $\underline{\mathbb{P}^N}(S)$ represented by the line bundle $\lambda_U(M)$ and the base-point free linear system $\langle \theta_{\gamma_0}(M), \ldots, \theta_{\gamma_N}(M) \rangle$ (cf. [26, Lecture 5] or [11, Ch II, Th 7.1]).

A similar remark applies to (7.13) below.

As a corollary of Theorem 7.8, using essentially the adjunction in (7.5), we obtain a similar result for sheaves, with the usual more delicate conditions on the embedding.

Theorem 7.10. *For any Hilbert polynomial P, we can find vector spaces U_0, U_1 satisfying* (7.10) *and finitely many maps*

$$\delta_0, \ldots, \delta_N \colon U_1 \otimes \mathcal{O}(-m) \to U_0 \otimes \mathcal{O}(-n)$$

such that the map

$$\Theta_\delta \colon \mathcal{M}_X^{ss} \to \mathbb{P}^N \colon [E] \mapsto (\theta_{\delta_0}(E) : \cdots : \theta_{\delta_N}(E)) \qquad (7.13)$$

is a closed set-theoretic embedding. This embedding is scheme-theoretic in characteristic zero, while in characteristic p it is scheme-theoretic on the stable locus.

Proof. We obtain the embedding, and its properties, by combining the embedding $\varphi \colon \mathcal{M}_X^{ss} \to \mathcal{M}_A^{ss}$ of Proposition 6.7 and the embedding $\Theta_\gamma \colon \mathcal{M}_A^{ss} \to \mathbb{P}^N$ of Theorem 7.8. To see that $\Theta_\delta = \Theta_\gamma \circ \varphi$, we need the observation from Proposition 7.7 that $\varphi^* \theta_{\gamma_i} = \theta_{\delta_i}$, for $\delta_i = \Phi^\vee(\gamma_i)$. $\square$

Remark 7.11. In characteristic zero, by considering the regularity of the ideal sheaf of the embedding $\mathcal{M}_X^{ss} \subset \mathcal{M}_A^{ss}$, we may deduce that, for sufficiently large U_0, U_1, the restriction map

$$\varphi^* \colon H^0(\mathcal{M}_A^{ss}(a,b), \lambda_U(a,b)) \to H^0(\mathcal{M}_X^{ss}(P), \lambda_U(P))$$

is surjective. Hence, as $H^0(\mathcal{M}_A^{ss}, \lambda_U)$ is always spanned by determinantal theta functions θ_γ, we deduce that $H^0(\mathcal{M}_X^{ss}, \lambda_U)$ is spanned by determinantal theta functions θ_δ, for sufficiently large U_0, U_1.

7.4. **Faltings' theta functions on curves.**

A result of Faltings gives the following cohomological characterisation of semistable bundles on a curve (see [6] in characteristic zero and [33] in arbitrary characteristic).

Theorem 7.12. *Let X be a smooth projective curve. A bundle E on X is semistable if and only if there exists a non-zero bundle F such that*

$$\operatorname{Hom}_X(F, E) = 0 = \operatorname{Ext}_X^1(F, E). \tag{7.14}$$

This condition (7.14) may be interpreted as saying that E and F are 'perpendicular' in the derived category $D^b(X)$, in the sense that $\mathbf{R}\operatorname{Hom}_X(F, E) = 0$. Furthermore, this has the immediate numerical consequence that

$$\chi(F, E) := \sum_{i \geq 0} (-1)^i \dim \operatorname{Ext}_X^i(F, E) = 0. \tag{7.15}$$

Just as in §7.1, Theorem 7.12 may also be interpreted as saying that certain determinantal theta functions detect the semistability of bundles on smooth curves.

To be precise about what this means, suppose that E is a family over S of bundles on X and F is a bundle such that $\chi(F, E) = 0$. Then $\mathbf{R}\operatorname{Hom}_X(F, E)$ is represented (locally over S) by a complex $d \colon \mathcal{K}^0 \to \mathcal{K}^1$ of vector bundles, of the same rank, such that, fibrewise at each $s \in S$, $\ker d_s = \operatorname{Hom}_X(F, E_s)$ and $\operatorname{coker} d_s = \operatorname{Ext}^1(F, E_s)$. There is then a well-defined line bundle λ_F defined globally on S, which is canonically isomorphic (locally) to $\det(\mathcal{K}^0)^{-1} \otimes \det(\mathcal{K}^1)$ and with a section θ_F canonically identified (locally) with $\det d$. Note (see e.g. [33, Lemma 2.5]) that, if $r(F_1) = r(F_2)$ and $\det F_1 = \det F_2$, then $\lambda_{F_1} = \lambda_{F_2}$, so that ratios between such theta functions θ_{F_1} and θ_{F_2} can meaningfully provide projective coordinates.

Indeed, Faltings [6] shows that it is possible to find finitely many $F_0, \ldots, F_N$ which detect all semistable bundles (of given rank r and degree d) and for which the morphism on the corresponding moduli space,

$$\Theta_F \colon \mathcal{M}_X^{ss}(r, d) \to \mathbb{P}^N \colon [E] \mapsto (\theta_{F_0}(E) : \cdots : \theta_{F_N}(E)),$$

is the normalisation of its image, thereby giving an implicit construction of the moduli space. Seshadri [33, Remark 6.1] asks how close this normalisation is to being an isomorphism, or indeed, how close the

theta functions θ_F come to spanning the space sections of the theta bundle λ_F on $\mathcal{M}_X^{ss}$. Esteves [5, Theorems 15,18] made progress by showing that one can find a Θ_F which is a set-theoretic embedding and which, in characteristic zero, is a scheme-theoretic embedding on the stable locus $\mathcal{M}_X^s$.

Now, using Theorem 7.10 and Remark 7.11, we can give a complete answer to Seshadri's question, at least in characteristic zero.

First note that, by placing some reasonable restrictions on E and F it is possible to define theta functions globally.

Lemma 7.13. *Suppose that E is a family over S of n-regular sheaves and that, for some F with $\chi(F, E) = 0$, there is a short exact sequence*

$$0 \to F' \xrightarrow{f} U \otimes \mathcal{O}(-n) \longrightarrow F \to 0. \tag{7.16}$$

Then $f^*\colon \operatorname{Hom}_X(U, E(n)) \to \operatorname{Hom}_X(F', E)$ is a map of vector bundles on S, of the same rank, and $\theta_F = \det f^*$.

Proof. For any $s \in S$, apply the functor $\operatorname{Hom}_X(-, E_s)$ to the short exact sequence (7.16). The resulting long exact sequence has just six terms, because X is a smooth curve. The fifth term $\operatorname{Ext}_X^1(U \otimes \mathcal{O}(-n), E_s)$ vanishes because E_s is n-regular. Hence the sixth term $\operatorname{Ext}_X^1(F', E_s)$ also vanishes.

This vanishing means that $\operatorname{Hom}_X(F', E_s)$ is the fibre of a vector bundle $\operatorname{Hom}_X(F', E)$ of rank $\chi(F', E)$ and that $\operatorname{Hom}_X(U, E_s(n))$ is the fibre of a vector bundle $\operatorname{Hom}_X(U, E(n))$ of rank $P(n) \dim U$, which is equal to $\chi(F', E)$, because $\chi(F, E) = 0$.

Now, the remainder of the long exact sequence shows that the map

$$(f^*)_s\colon \operatorname{Hom}_X(U, E_s(n)) \longrightarrow \operatorname{Hom}_X(F', E_s)$$

has kernel $\operatorname{Hom}_X(F, E_s)$ and cokernel $\operatorname{Ext}_X^1(F, E_s)$, so that f^* represents $\mathbf{R}\operatorname{Hom}_X(F, E)$ (globally) and hence $\theta_F = \det f^*$, as required. $\square$

Using this we can show that the determinantal theta functions θ_δ from §7.1 are also theta functions in the sense of Faltings.

Proposition 7.14. *Let $\delta\colon U_1 \otimes \mathcal{O}(-m) \to U_0 \otimes \mathcal{O}(-n)$ be such that $\operatorname{Hom}_X(\delta, E_0)$ is invertible for some bundle E_0 of rank r and degree d and let $F = \operatorname{coker} \delta$.*

Then $\lambda_U \cong \lambda_F$ and $\theta_\delta = \theta_F$, on any family of n-regular sheaves with rank r and degree d. In particular, for E in such a family, $\operatorname{Hom}_X(\delta, E)$ is invertible if and only if $\mathbf{R}\operatorname{Hom}_X(F, E) = 0$.

Proof. We now have two short exact sequences

$$0 \to F' \xrightarrow{f} U_0 \otimes \mathcal{O}(-n) \longrightarrow F \to 0, \tag{7.17}$$

$$0 \to F'' \longrightarrow U_1 \otimes \mathcal{O}(-m) \xrightarrow{g} F' \to 0, \tag{7.18}$$

where $F' = \operatorname{im} \delta$, $F'' = \ker \delta$ and $\delta = fg$.

Note that, because X is a smooth curve and hence its category of coherent sheaves has global dimension 1, we know that δ, regarded as a complex $K_\bullet$, is quasi-isomorphic to the direct sum of its cokernel and its (shifted) kernel. Thus, $\mathrm{Hom}_X(\delta, E)$ is invertible if and only if $\mathbf{R}\,\mathrm{Hom}(K_\bullet, E) = 0$ (see Remark 7.4), which in turn is if and only if $\mathbf{R}\,\mathrm{Hom}(F, E) = 0$ and $\mathbf{R}\,\mathrm{Hom}(F'', E) = 0$. Because we are assuming that this happens for one bundle E_0 of rank r and degree d, this implies, in particular, that $\chi(F, E) = 0 = \chi(F'', E)$, for any sheaf E of the same rank and degree, by Riemann-Roch.

Now, we also observe that, for any E, we have the following factorisation of $\mathrm{Hom}_X(\delta, E)$, written here as δ^*.

$$
\begin{array}{c}
\mathrm{Hom}_X(U_0, E(n)) \\
{\scriptstyle f^*}\Big\downarrow \searrow {\scriptstyle \delta^*} \\
\mathrm{Hom}_X(F', E) \xrightarrow{ g^* } \mathrm{Hom}_X(U_1, E(m))
\end{array}
\qquad (7.19)
$$

The horizontal map g^* is always injective. Thus, to prove the equality of theta functions, what we need to show is that, when E is n-regular, g^* is an isomorphism, so that $\lambda_U \cong \lambda_F$ and

$$
\theta_\delta = \det \delta^* = \det f^* = \theta_F,
$$

where the last equality is by Lemma 7.13.

From the long exact sequence obtained by applying $\mathrm{Hom}_X(-, E)$ to (7.18) we see that it is sufficient to show that $\mathrm{Hom}_X(F'', E) = 0$. We also see from the same long exact sequence that, when E is n-regular, $\mathrm{Ext}^1_X(F'', E) = 0$ and so the result follows, because we know that $\chi(F'', E) = 0$. $\qquad\square$

Note that the F that occur here are necessarily vector bundles.

Corollary 7.15. *For given r, d, there exists a finite set $F_0, \ldots, F_N$ of vector bundles such that the map*

$$
\Theta_F \colon \mathcal{M}_X^{ss}(r, d) \to \mathbb{P}^N \colon [E] \mapsto (\theta_{F_0}(E) : \cdots : \theta_{F_N}(E))
$$

is a closed set-theoretic embedding. This embedding is scheme-theoretic in characteristic zero, while in characteristic p it is scheme-theoretic on the stable locus.

Proof. Immediate from Theorem 7.10 and Proposition 7.14. $\qquad\square$

Thus, in characteristic zero, we see that Faltings' determinantal theta functions can be used to give projective embeddings of the moduli spaces of semistable bundles on a smooth curve. Furthermore, by Remark 7.11, we have a positive answer to Seshadri's question: the theta functions θ_F span the sections of line bundles λ_U of sufficiently high degree.

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CSIC, Serrano 113 bis, 28006 Madrid, Spain
E-mail address: lac@mat.csic.es

Mathematical Sciences, University of Bath, Bath BA2 7AY, UK
E-mail address: a.d.king@maths.bath.ac.uk